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TANTON'S TAKE ON …

DOES "AND" MEAN MULTIPLY IN PROBABILITY?

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The following idea feels right in basic probability theory:

If a certain outcome of an experiment has a probability $p\%$ of occurring, then in performing the experiment many, many times, we'd expect to see that outcome about $p\%$ of the time.

For example, if I roll a die a million times, I'll likely see a roll of 5 about one-sixth of the time. If I toss a coin a $90,000$ times, I'd likely see about $45,000$ heads. And so on.

This idea also works in reverse. For example, if I toss a coin 500 times and it comes up HEADS 403 times, I'd strongly suspect that the coin is biased (biased with a probability of about 80% for tossing a head.)

Imagining repeating an experiment a large number times can help you think through a challenging probability problem:

EXAMPLE: A bag contains 8 Tuscan sunset orange balls and 2 Tahitian sunrise orange balls. The color difference is very subtle and only 70% of people can correctly identify the color of a ball when handed one. Lulu pulls a ball out of the bag at random and tells you over the phone that she pulled out a Tahitian sunrise ball. What are the chances that the ball she holds really is Tahitian sunrise?

Answer: Let's assume, as the statistic given suggests, that there is a 70% chance that Lulu can correctly identify the color of a ball when handed one.

Now imagine Lulu conducting this ball-picking experiment a large number of times, say 100 times (and that her chances of correctly identifying colors does not change.)

About 80 of the balls Lulu pulls out will be Tuscan sunset. Of those, she'll identify about $0.7 \times 80 = 56$ of them correctly as Tuscan sunset and 24 she'll incorrectly say are Tahitian sunrise.

About 20 of the balls Lulu pulls out will be Tahitian sunrise, of which $0.7 \times 20 = 14$ she'll correctly identify as such. However, she'll call 6 of them Tuscan sunset.

56 Tuscan sunset Says Tuscan sunset	14 Tahitian sunrise Says Tahitian sunrise
24 Tuscan sunset Says Tahitian sunrise	6 Tahitian sunrise Says Tuscan sunset

Thus in these 100 runs of the experiment we see that Lulu will say "Tahitian sunrise" about $24 + 14 = 38$ times and will be correct in saying this 14 of those times. This shows that the probability of her ball really being Tahitian sunrise is

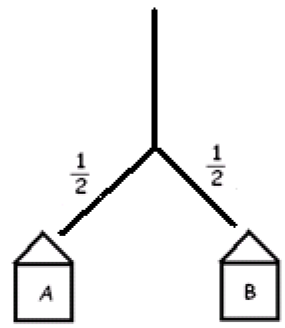
$$\frac{14}{38} \approx 37\%,$$

pretty low!

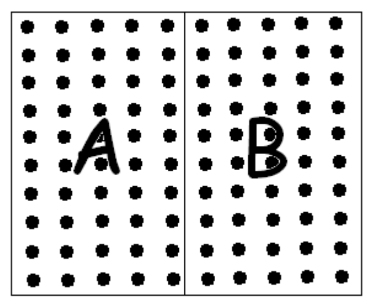
Comment: Even though eye-witnesses can identify suspects with reasonably high certainty, the probabilities that they are actually correct in their claims can still be low!

GARDEN PATHS

Suppose 100 people walk down a garden path that leads to a fork. Those who turn left go to house A, those who turn right to house B. Assume that there is a 50% chance that a person will turn one way over the other.



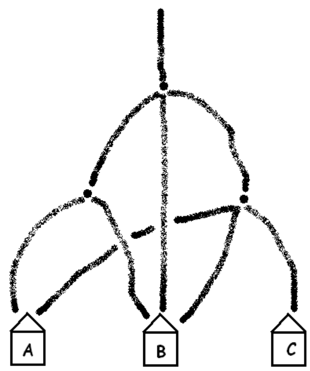
In this set-up we'd expect, essentially, 50 people to end up at house A and 50 people at house B. The following diagram of one-hundred dots (for 100 people) depicts this outcome:



The number "100" here is immaterial. The point is that if a square is used to denote the entire population of people walking down the path, then half the area of the square (half the people) end up with result A, and the second half of the square with result B.

Playing with examples like these is fun!

EXAMPLE: *Folk walk down the following system of paths. Use the square model to compute the fraction of people that end up at house A, at house B, and at house C. (Assume that each choice encountered at a fork in the path is equally likely.)*



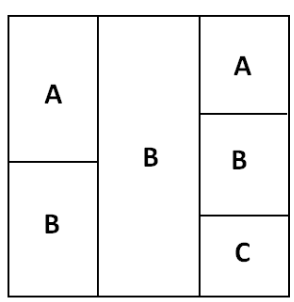
Answer: At the first fork, a third of the people turn left, a third go straight, and a third turn right.

Of those that turn left, half go to house A and half go to house B.

All those who go straight, go to house B.

Of those who turn right, a third go to A, a third to B, and a third to C.

We have:



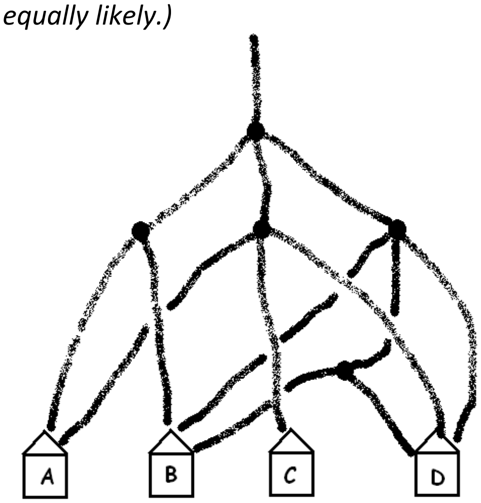
We see now that the proportion of people that end up in house A is given as half of a third, that's $\frac{1}{6}$, plus a third of a third, that's $\frac{1}{9}$. This proportion is $\frac{1}{6}+\frac{1}{9}=\frac{5}{18}$.

The proportion of people that end up in house B is: $\frac{1}{6}+\frac{1}{3}+\frac{1}{9}=\frac{11}{18}$.

The proportion of people that end up in house C is: $\frac{1}{9}=\frac{2}{18}$.

(**Question**: Does it make sense that these three answers add to 1?)

PRACTICE: *People walk down the following system of paths. Use the square model to compute the fraction of people that end up at each house. (Again assume that each choice encountered at a fork in the path is equally likely.)*

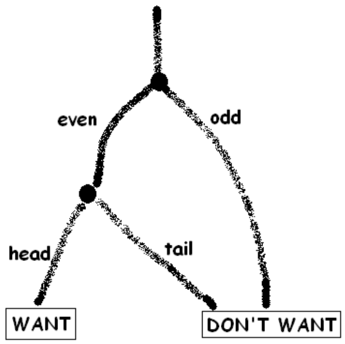


(Just so you have it, the answers are: $\frac{5}{18}$, $\frac{6}{18}$, $\frac{2}{18}$, and $\frac{5}{18}$.)

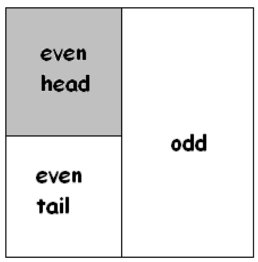
Of course, in sending people down these garden paths, we're really looking at the outcome of performing a probability experiment a large number of times.

EXAMPLE: *I roll a die and then toss a coin. What are the chances of getting an even number followed by a head?*

Answer: Think of this as a path-walking problem with two houses labeled "WANT" and "DON'T WANT." The forks in the road represent the options that can occur (each, with 50% chance of occurring):



This leads to the square model diagram:

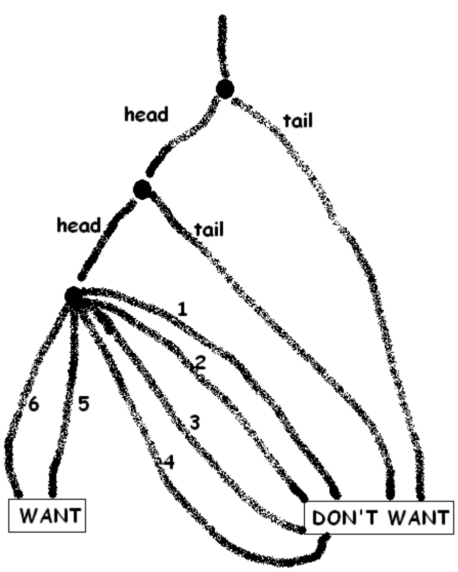


We see that the desired outcome represents one quarter (half of a half) of the square. Thus

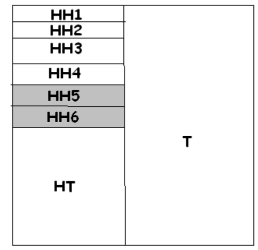
$$p\left(even\ \ AND\ \ head\right)=\frac{1}{4}.$$

EXAMPLE: *I toss a quarter, then I toss a dime, and then I roll a die. What are the chances of receiving HEAD, HEAD, and "5 or 6"?*

Answer: Here's the garden path:



This gives the square model:



We have:

$$p\left(head\ \ AND\ \ head\ \ AND\ \{5,6\}\right)$$

$$=\frac{2}{6}\text{ of }\frac{1}{2}\text{ of }\frac{1}{2}\text{ of the square}$$

$$=\frac{2}{6}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{12}.$$

Notice that each of the fractions in the final answer is a fraction corresponding to the probability of an individual outcome in the series of tasks: $\frac{2}{6}$ is the chance of rolling a five or a six with a die, $\frac{1}{2}$ is the chance of flipping a head with a dime, and $\frac{1}{2}$ is the chance of flipping a head with a quarter.

We see we get these fractions of a fraction of a fraction of a square – the product of these fractions.

In general:

MULTIPLICATION PRINCIPLE IN PROBABILITY THEORY:

If one performs one task and hopes to get outcome A, and then performs a second, completely unrelated, task and hopes to get outcome B, then the probability of seeing A and then B is the product of the individual probabilities:

$$p\left(A\ and\ then\ B\right)=p\left(A\right)\times p\left(B\right).$$

The garden path model shows this: $p\left(A\ and\ then\ B\right)$ is a fraction of a fraction of the area of a square.

"AND" MEANS MULTIPLY?

The multiplication principle stated here relies on actions being independent, yet we often teach students to use this principle any time we use the word "and" in a probability problem, even if the events involved are NOT independent. I often hear the aphorism *"And means multiply"* bandied about in a care-free manner. Does "and" always correspond to multiplication?

Consider the following two problems:

EXAMPLE 1: (A "with replacement" problem.) *A bag contains two red balls and three yellow balls. I pull out a ball at random, note its color, put the ball back, and pull out a second ball at random. What are the chances I see two red balls?*

EXAMPLE 1: (A "without replacement" problem.) *A bag contains two red balls and three yellow balls. I pull out a ball at random, note its color, and put it aside. I then pull out a second ball at random from the four balls that remain in the bag. What are the chances I see two red balls?*

The two actions in Example 1 are certainly independent (in no way does the outcome of pulling out a ball the first time affect the outcome of pulling out a ball a second time), and so by the multiplication principle we have:

$$P\left(RED\, and\ \ RED\right) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}.$$

But the two events in the Example 2 are not independent: the result of the first action affects the possible results of the second. (The probability of choosing a red second ball is either $1/4$ or $2/4$, depending on the outcome of the first action.)

If we blindly follow the aphorism *"And means multiply,"* then I am confused.

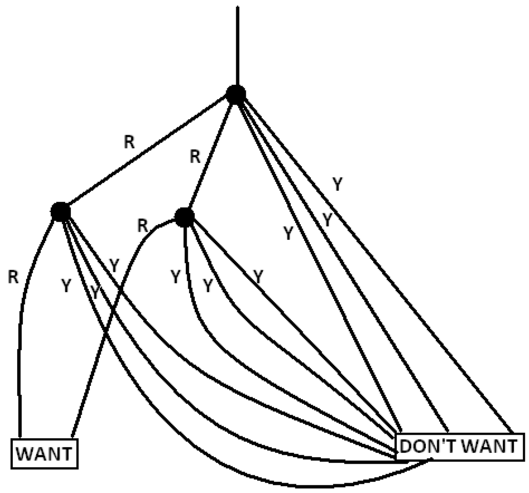
Is:

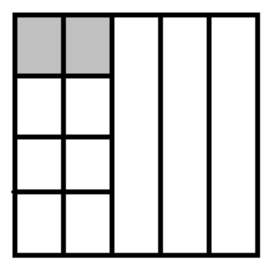
$$P\left(RED\, and\ \ RED\right) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$$

or is

$$P\left(RED\, and\ \ RED\right) = \frac{2}{5} \times \frac{2}{4} = \frac{4}{20}\ ?$$

A good way out of these pickles is to draw a garden path for the problem. Here's a diagram for Example 2. (Notice there is one path for each ball in the bag.)





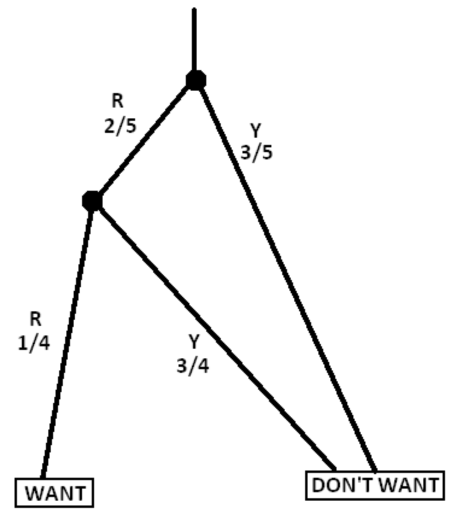
We see:

$$P\left(RED\, and\ \ RED\right) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{2}{20}$$

DONE! (The garden paths will not let you down. There is no need for anything more!)

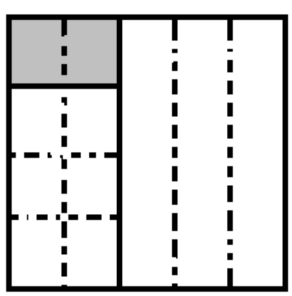
But for the purposes of this essay, we want to ask: *Was a multiplication principle at play here?* We have the addition and the multiplication of fractions in our answer.

In looking at an example like the one just given in a class conversation someone will invariably comment that we could have drawn a simplified version of the garden path model. GREAT! (Let students suggest these ideas!)



In this diagram, directions from a fork are no longer given equal weight: the probabilities of making a particular turn match the probabilities of the outcomes from the event the fork represents.

Now from this perspective the area model shows very directly a product of fractions:



One quarter of two-fifths.

We have multiplication:

$$P(RED\ and\ RED) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20}.$$

So it seems that "and" does correspond to the action of multiplication from this perspective.

For those who like general statements of principles (one can always just "nut things out" with garden paths), here it is:

MULTIPLICATION PRINCIPLE IN PROBABILITY THEORY:

Suppose one performs one task and hopes to get outcome A, and then performs a second task and hopes to get outcome B. To work out the probability of seeing A and then B, first work out:

$P(A)$, the probability of seeing A.

$P(B \mid A)$, the probability of seeing B under the assumption you have just seen A.

Then:

$$p(A\ and\ then\ B) = p(A) \times p(B \mid A).$$

NOTE: If the two events are independent, that is, the outcome of the first task in no way affects the outcome of the second, then $p(B \mid A) = p(B)$. (The chances of seeing B are irrelevant to whether or not we've just seen A.) In this case,

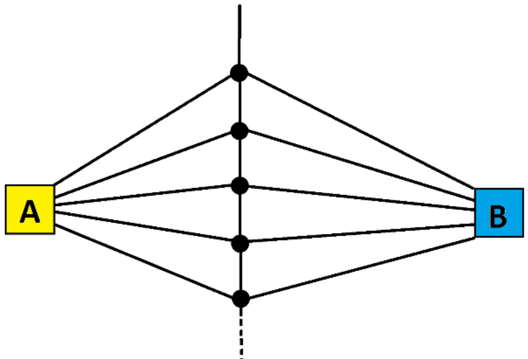
$$p(A\ and\ then\ B) = p(A) \times p(B)$$

and we have the multiplication principle we first described.

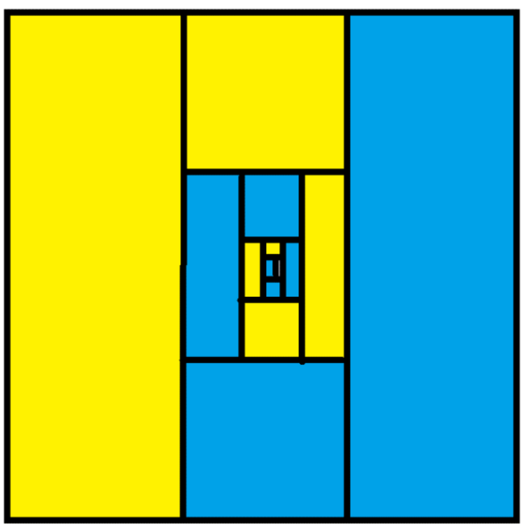
Comment: Garden paths are really much more fun!

SPEAKING OF FUN …

Imagine a garden path with infinitely many forks that split into three parts as shown. Those that turn left at a fork go to house A, those that turn right to house B, and those who go straight continue to the next fork.



This leads to the area picture:



The probability for ending up in house A is:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots,$$

as are the chances of ending up in house B.

But do you see that half the square is devoted to house A and half to house B? It follows then that we must have:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}.$$

CHALLENGE: Use garden paths to establish that

$$\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots=\frac{1}{N-1}$$

for any positive integer $N>1$.

Can you prove the general geometric series formula $x+x^2+x^3+\cdots=\frac{x}{1-x}$, for $0<x<1$, using garden paths?

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