CURRICULUM INSPIRATIONS: www.maa.org/ci

www.theglobalmathproject.org

INNOVATIVE CURRICULUM ONLINE EXPERIENCES: www.gdaymath.com
TANTON TIDBITS: www.jamestanton.com

TANTON'S TAKE ON …

EXPLODING DOTS as a MAKE SPACE

OCTOBER 2017

When I teach I tend to be the "sage on the stage." I am aware of this. But I strive to be an (alleged) sage that serves to let the mathematics discussed just shine for itself, thereby making my stage presence immaterial. My role is to foster an intellectual conversation between the mathematics and the individuals in the room with me simply acting as facilitator and perhaps Sherpa for that wonderful journey of wonder, discovery, and learning. I don't know if I succeed in guiding and fostering rather than simply delivering, but I do try step out of the way of the mathematics.

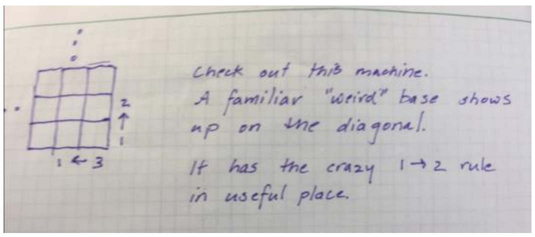
The rollout topic for our inaugural Global Math Week this month is *Exploding Dots*. It is mathematics, even if just delivered, that immediately invites invention, creativity, and play in and of itself. Like all true, joyous math thinking and doing, the content serves as its own creative MAKE SPACE.

As the world plays with *Exploding Dots* this month during Global Math Week, and in all the weeks and months that follow, I truly hope youngsters and adults alike will play and invent and share their discoveries and play with the world. Contact me and I'll share material here http://gdaymath.com/lessons/explodingdots/10-1-invent-create-enjoy/ as well as on the GMP community page.

Here are just a few snippets that already show the surprising and creative places folk can go with Exploding Dots.

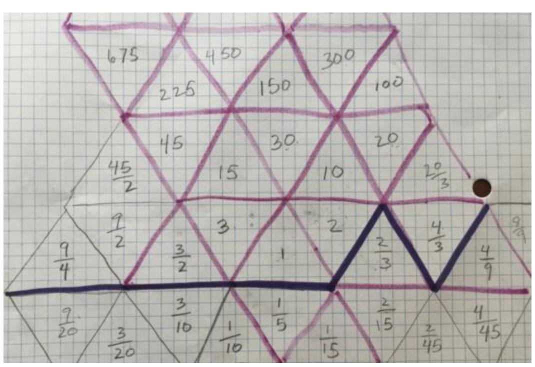
TWO-DIMENSIONAL MACHINES

Youngsters Goldfish & Robin from Eugene, OR, combine two *Exploding Dots* machines to make an exploding array. They see the curious $2 \leftarrow 3$ machine appear on the diagonal!



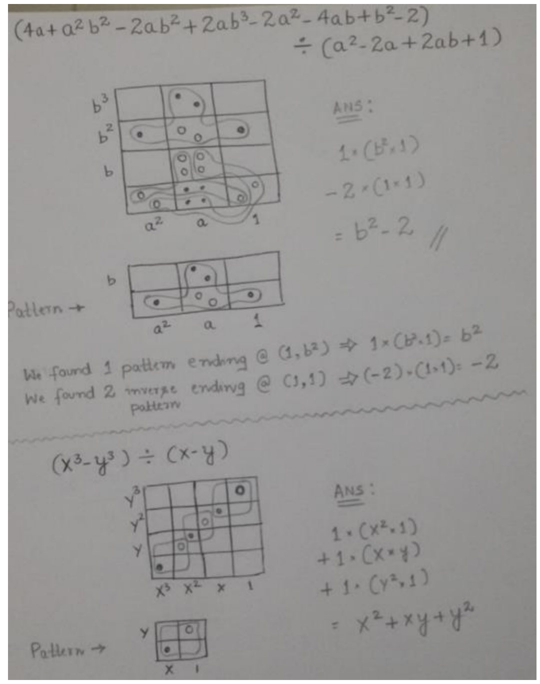
They really push matters all the way in this fabulous video: https://www.youtube.com/watch?v=0CTjkoFvGw0&feature=youtu.be

And why stay with square arrays?



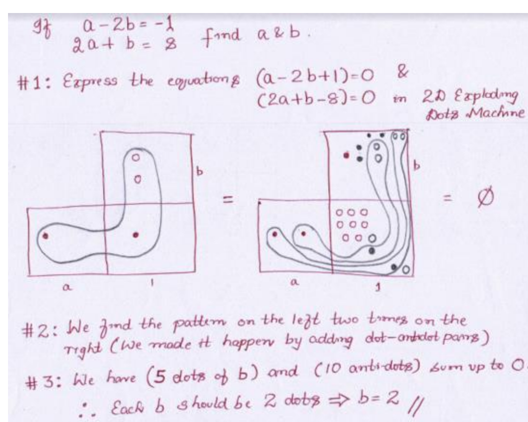
TWO-VARIABLE POLYNONIAL DIVISION

Global Math Project Kiran Bacce from realized that two-variable polynomial division is just as doable, and as fun, as one-variable polynomial division. It to uses two dimensions.



Kiran wrote about his ideas here: http://kbacche.weebly.com/uploads/9/6/0/4/96043016/exploding_dots_in_two_dimensions.pdf

He's also been playing with the idea of solving simultaneous equations this way.



ROMAN NUMERALS

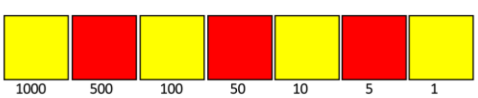
Some mathematical historians say that European mathematics was held back for many centuries because of the cumbersome Roman numeral system. It is a system that relies on different symbols for values of quantities (I = one, V = five, X = ten, L = fifty, and so on) and just simply groups symbols together that add to the desired value. For example, seventy-eight was represented LXXVIII.

Although the symbols were usually arranged from largest to smallest, left to right, there is no real notion of place-value here, and doing arithmetic directly with Roman numerals is nigh-on impossible.

$$\begin{array}{rr} & \text{CLXXVI} \\ + & \text{XXXVIII} \\ + & \text{MXI} \\ + & \text{XXV} \\ \hline = & \end{array}$$

Or is it?

Kiran Bacche suggests interleaving a $1 \leftarrow 10$ and a $1 \leftarrow 5$ machine to makes sense of and do arithmetic with Roman numerals.



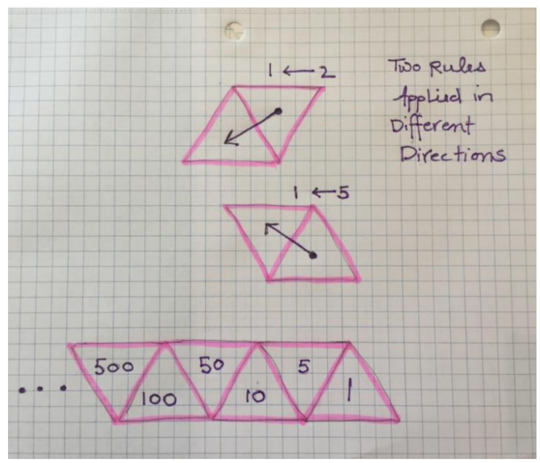
At some point, medieval Europe adopted a "subtraction principle" in writing Roman numerals: Placing a symbol of lesser value immediately to the left of a symbol of greater value is to be interpreted as the difference of those two values. For example,

IV = four
XC = ninety
CDXCVI = four hundred ninety six.

Kiran suggests it is even possible to make good sense of this subtraction principle too with dots (and anti-dots!)

Read his thoughts here: http://kbacche.weebly.com/uploads/9/6/0/4/96043016/interleaved_exploding_dots_and_roman_numerals.pdf

And sometimes people discover the same mathematics in multiple ways. Goldfish & Robin saw that their triangular arrays were doing the same interweaving.



PALINDROME SEARCH

The number 17 is a palindrome in a $2 \leftarrow 3$ machine. It has code 21012. And 494 seems to be the largest number known to be a palindrome in this machine.

Dr. Gary Davis explains here http://www.crikeymath.com/2017/09/16/implementation-of-the-exploding-dots-representation-algorithm/ how he hunts for palindromes. And Dr. James Propp writes extensively about this work in his beautiful Mathematical Enchantments essay here: https://mathenchant.wordpress.com/2017/09/17/how-do-you-write-one-hundred-in-base-32/#more-1835.

Can you find another $2 \leftarrow 3$ machine palindrome?

SURPRISING CONNECTIONS

After reading about polynomial division the *Exploding Dots* way, mathematics Professor Richard Hoshino, author of *The Math Olympian*, shared the following:

Inspired by what I read today, I spent a few hours at a coffee shop playing with Exploding Dots. From this super-creative session, I wanted to share two fun problems that you and your students might enjoy based on the last chapter of James' text.

1. James defines a $1|0|0 \leftarrow 0|1|1$ machine and explains why this is equivalent to a $1 \leftarrow \varphi$ machine, where φ is the golden ratio $\frac{1+\sqrt{5}}{2}$. I was interested in seeing what positive integers had "machine" representations with as few 1s as possible, and which positive integers had representations with as many 1s as possible. Let $f(n)$ be the "phi-nary" representation of n. I noticed a particularly shocking pattern:

f(2) = 10.01,
f(3) = 100.01,
f(7) = 10000.0001,
f(18) = 1000000.000001,
f(47) = 100000000.00000001

f(1) = 1,
f(4) = 11.1111,
f(11) = 1111.111111,
f(29) = 111111.11111111,
f(76) = 11111111.1111111111

Let's look at all the integers n for which $f(n)$ is either all 1's, or consists of only 1s at the ends. We get the sequence {2, 1, 3, 4, 7, 11, 18, 29, 47, 76,...}. This is known as the Lucas sequence, i.e., the Fibonacci sequence with starting terms 2 and 1. Beautiful, eh?

Question: prove that n has one of these two special "phi-nary" representations if and only if n is a Lucas number.

2. In 1996, as a high school senior, I wrote the United States of America Mathematical Olympiad (USAMO), arguably a harder math contest than the International Mathematical Olympiad (IMO). The last question of the 1996 paper is as follows:

Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a+2b=n$ with a and b both belonging to set X.

The majority of us (me included) got shut out on this problem, since it was hard to make progress on this problem. One idea is to attempt to construct some set X that satisfies the property, starting with small negative and positive numbers. But it's hard to make progress, given that one needs to rigorously justify that any integer n can be represented in the form $a+2b$ in exactly one way.

Here is where the Exploding Dots machine comes to the rescue!

Consider a $-1 \leftarrow 4$ machine, or what we would think of as "base negative four". For example,

$$111 = (-4)^2 + (-4)^1 + (-4)^0 = 13.$$

Let X be the set of integers whose machine representations in "base negative four" consist only of 0s and 1s, with no 2s and 3s. For example, 13 belongs to set X, but 14 does not.

Question: prove that X is the desired set in the above USAMO question, i.e., for any integer n, there is exactly one way that n can be written as the sum $a+2b$, where a and b both have only 0s and 1s in every box of the $-1 \leftarrow 4$ Exploding Dots Machine.

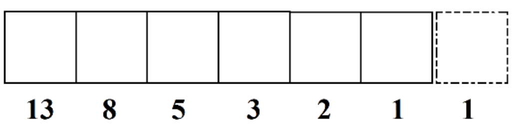
ACTUALLY BUILD A MACHINE!

Dr. Glen Whitney of mathwalks.org did. See a $2 \leftarrow 3$ machine in action here: https://www.youtube.com/watch?v=1_jJQu1px-8.

FIBONACCI PLAY

I've been playing with a machine that has the rule: two dots in neighboring boxes are replaced with one dot, one place to their left.

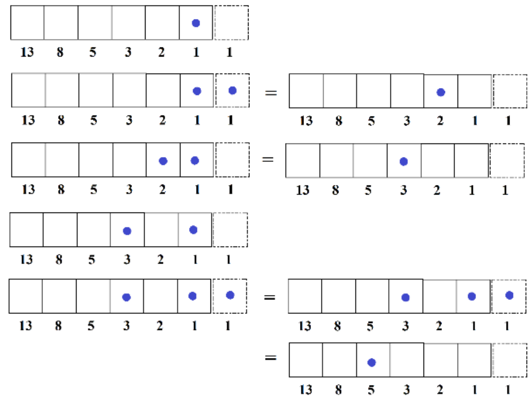
$$100 \longleftarrow 011$$



It happens to be closely related to a $1 \leftarrow \varphi$ Dr. Hoshino plays with above, except I have two "first" boxes. Box values match the Fibonacci numbers precisely: any two consecutive Fibonacci numbers sum to the third.

As you put dots into the machine one at a time, dots always go in the solid box labeled 1, unless it is full, in which case it goes in the dashed box label 1.

Here the results of placing one to six dots into the machine.

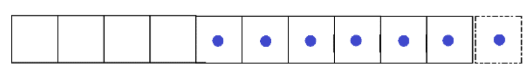


This machine shows that every number can be expressed as a sum of distinct Fibonacci numbers, no two being consecutive! This is a famous observation due to Edouard Zeckendorff.

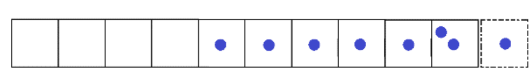
Actually one can prove such a representation of any number is sure to be unique.

Zeckendorff's Theorem: *Each and every counting number can be written, uniquely, as a sum of distinct Fibonacci numbers with no two being consecutive Fibonacci numbers.*

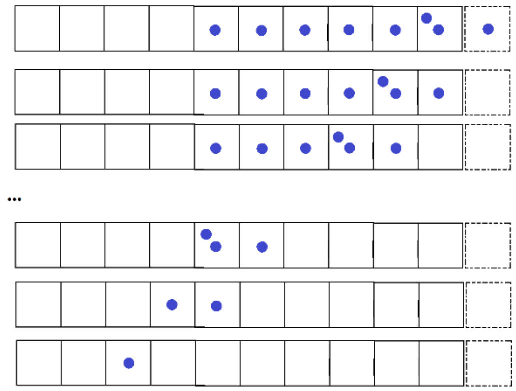
One can have fun with this machine to prove all sorts of results about the Fibonacci numbers. For example, this "fully loaded" machine represents the sum of the first nine Fibonacci numbers $F_1 = 1$ through to $F_9 = 34$.



Before we do any explosions, let's add one more dot to the machine.



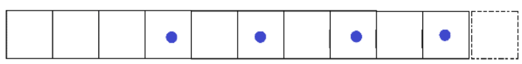
Now let's explode!



We see this equals the 11th Fibonacci number. In general, we have

$$F_1 + F_2 + F_3 + \cdots + F_N + 1 = F_{N+2}.$$

Adding one more dot to this picture



shows that

$$F_2 + F_4 + F_6 + \cdots + F_{2N} + 1 = F_{2N+1}.$$

And can you now prove that

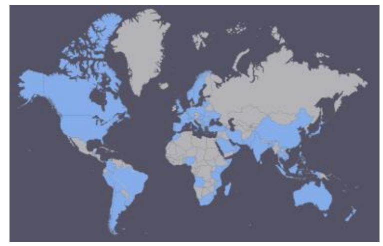
$$F_1 + F_3 + F_5 + \cdots + F_{2N-1} = F_{2N} ?$$

Further: Can you in fact prove that any sum of distinct Fibonacci numbers smaller than F_N containing no two consecutive Fibonacci numbers is sure to be less than F_{N+1}? Can you use this observation to now prove that the Zeckendorff representations of numbers are indeed unique?

GLOBALMATHPROJECT

Global Math Week is Oct 10-17. Our goal is to simply have the whole world take part in a common piece of joyous, uplifting, classroom-relevant piece of mathematics some time during that special week. Modest, eh?

The rollout topic for this year is *Exploding Dots*. See www.theglobalmathproject.org for details.



Already hundreds of thousands of teachers and students from over 90 different countries have signed on. And it is easy for you to join in too.

Here's the four-step process:

1. **Experience *Exploding Dots* for yourself.**
   See our site.

2. **Register at our site.**
   Have you and your students count towards this global phenomenon!

3. **Do some *Exploding Dots* with your students during Global Math Week.**
   One class period. Half a class period. Even 15-minutes will count! See the teaching guides on our site.

4. **Share comments, photos, videos with the world on social media.**
   Be part of the global community.

GLOBAL MATH PROJECT on social media:

Twitter: @globalmathproj
#gmw2017
#explodingdots
www.facebook.com/theglobalmathproject

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