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TANTON'S TAKE ON …

WHAT IS A FUNCTION?

OCTOBER 2016

When it comes to functions, the high school curriculum is primarily focused on ones that transform real numbers into other real numbers. But there is something of value in first introducing the function concept, and all the key subsidiary concepts, without a primary focus on number. It all feels quite natural and straightforward.

And I am not alone in this viewpoint (though I may take the idea a little further than others). Writers of your State Standards, for example, have subtly introduced the function concept in middle-school geometry by describing a geometric transformation as "rule" that assigns to each point in the plane a new point in the pane, called its image point. The hope is that high-school curriculum writers will pick up on this familiarity when they formally introduce the notion of a function.

Historical Comment: Work on formalizing this idea of a transformation began in the 1600s. Around 1675, calculus scholar Gottfried Wilhelm Leibniz coined the word "function" for a rule that transforms objects from some set into new objects, basing the name on the Latin word *fungi* meaning "to perform." (This is not to be confused with the Greek word for "sponge" which led to the modern botanical word *fungus*.) His thinking and work was popularized by Leonhard Euler, who is credited as the first to use the notation "$f(x)$" so familiar to us today.

FUNCTIONS: A VERY SWIFT OVERVIEW OF EVERYTHING ABOUT THEM

Suppose we have two sets of objects X and Y.

Loosely speaking, a *function* from X to Y is a rule, supposedly clearly understood by all, which assigns to each and every object from the set X precisely one object from the set Y.

Example: If X is the set of American citizens and Y is the set of nine-digit numbers, then the rule that assigns to each citizen a social security number is a function.

Example: Let X be the set of points in the plane and Y be the same. Then a given rotation is a function: it assigns to each point its image point.

If we use the letter f, say, to denote the function, then we write

$$f : X \to Y.$$

We usually call each element x of X an *input* of the function and denote the element assigned to it from Y by $f(x)$. This is called the matching *output* for the input x.

Comment: $f(x)$ is read out loud as "f of x."

Comment: Actually, it is not at all common in the school world to write "$f : X \to Y$."

Example: Consider the rule, which we shall denote M, that assigns to each person his or her biological mother. This is a well-defined rule from the set of all people of the world to the set of all women, and so we have a function.

$$M : \text{People} \to \text{Women}.$$

My name is James and mother's name is Abby and so we have, for instance,

$$M(\text{James}) = \text{Abby},$$

which reads "the mother of James is Abby."

(Although there are many James' and Abbys of the world, it is assumed here that we are referring to the specific people I have in my mind right now, namely, myself and my mother.)

Thinking of more people in my family, we have $M(\text{Turner}) = \text{Lindy}$ and $M(\text{Lindy}) = \text{Sally}$. We also have $M(M(\text{Turner})) = \text{Sally}$ to be read as "the mother of the mother of Turner is Sally."

And this rule is indeed a function: each and every person is "assigned" one, and only one, biological mother. That two or people can be assigned the same mother does not violate the definition of being a function, nor does the fact that not all women are mothers.

Example: Consider the rule

$$A : \text{People} \to \text{Natural Numbers}$$

that assigns to each living person his or her age expressed as a whole number of years.

This is a function, and at the time of writing this piece we have $A(\text{James}) = 50$. That there are many people the same age as me does not violate the definition of this rule being a function.

Example: Consider the "rule"

$$D : \{\text{Men over the age of } 40\} \to \text{People}$$

which assigns to each man over the age of 40 his biological daughter.

This is not a function as the rule has two problems: not every man has a daughter, and some men have more than one daughter (which do we assign?). We say that this rule is not *well defined*.

We could salvage the previous example by restricting D to the set of all men over the age of 40 who have at least person in his life which he regards as a daughter, and assign to each such fellow his youngest emotional/biological daughter.

Example: Consider the truth function

$$T : \text{The set of all possible statements} \to \{\text{True}, \text{False}\}$$

given by the rule: Assign the word "true" to a statement if the statement is true. Assign the word "false" to a statement if the statement is false.

For example,

$$T(\text{Liquid water is wet}) = \text{True}$$
$$T(\text{James likes cooked green peppers}) = \text{False}$$
$$T(\text{This sentence is true}) = ?$$

This example shows that our loose definition of a function as a "rule" is problematic. What constitutes a meaningful rule?

AN ATTEMPT AT A FORMAL DEFINITION OF A FUNCTION

Rather than think of a function as defined as a "rule," one can instead, in an attempt to be formal, define a function to simply be a set, a set of special pairs constructed from two given sets. In this – somewhat dry – thinking, a function f from a set X to a set Y is a collection of ordered pairs (x, y), with x an element of X and y an element of Y, with the property that for each x in X there is precisely one pair in the collection f of pairs with first element x.

For example, if $X = \{1, 2, 3\}$ and $Y = \{A, B\}$, then the collection of pairs

$$f = \{(1, A),\ (2, C),\ (3, A)\}$$

is a function, but the collections

$$g = \{(1, A), (1, B), (2, A), (3, C)\}$$

and

$$h = \{(2, B),\ (3, A)\}$$

are not functions from X to Y.

Comment: One can attempt to be even more formal, and hence dryer and less intuitive, and say:

For a pair of sets X and Y, let $X \times Y$ denote the set of all possible ordered pairs (x, y) with x from X and y from Y. Then a function f from X to Y is a subset of $X \times Y$ with the property that each and every element x of X appears as a first element of exactly one pair in f.

To mathematicians being formal, a function is then just a special subset of $X \times Y$. This gives the feel of being clear and precise, but it is actually just as problematic as our loose opening definition: how do you describe which pairs (x, y) belong to the special subset? For example, if X is the set of all the people of the world and Y the set of all men, then one still needs to say something like "(x, y) belongs to the father function only if y is x's biological father."

Question: If X and Y are sets, is it possible for the entire set of pairs $X \times Y$ to be a function?

Quick Answer: Yes, if Y is a set with only one element.

Question: Suppose a function assigns to each element x of a set X the element x. (This is called the *identity* function. It does nothing to inputs.) Describe the subset of $X \times X$ that defines this function.

Quick Answer: We have the "diagonal subset," the set of all pairs of the form (x, x) with x in X.

Question: If A is a set with a elements in it, and B as set with b elements, how many different functions f are there $f : A \to B$?

Quick Answer: There are b^a functions.

SOME JARGON

The set of all allowable inputs for of a function is called the *domain* of the function. The set of all possible outputs is called its *range*.

Comment: If we write $f : X \to Y$, then it is implied that the domain of the function is X. The range of the function, however, need not be all of Y. For example, for the age function described above

$$A : \text{People} \to \text{ Natural Numbers}$$

the domain of the function is the set of all people of the world and the range is the set of whole numbers $\{0, 1, 2, 3, ..., 120(?)\}$. (How old is the oldest current living person?)

Example: Consider the function $F : \text{Counting Numbers} \to \text{Counting Numbers}$ which assigns to each counting number its first digit. (So $F(902) = 9$ and $F(8) = 8$.) Here the domain is the set of all counting numbers and the range is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

COMPOSTION OF FUNCTIONS

One can imagine a function to be a "machine" that takes inputs and churns each into an output.

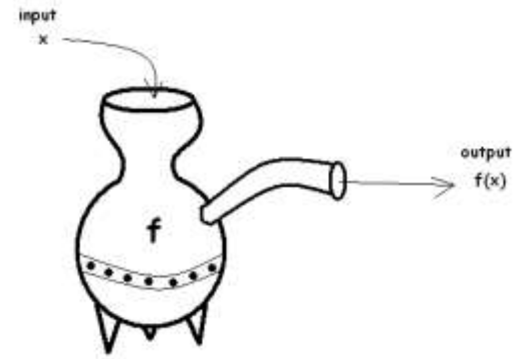

Figure 1.

As such, we can imagine linking together two, or more, machines.

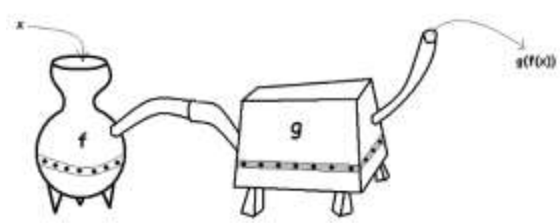

Figure 2.

Figure 2 assumes that each output of a function $f : X \to Y$ is a valid input of a function $g : Y \to Z$. The action of these two functions together take an element x of X and first "hits" it with f to obtain $f(x)$ and then "hits" this output with g to give $g(f(x))$.

Example: Consider the father function F which assigns to each living person his or her biological father and the year of birth function Y which assigns to each person the year he or she was born. Then $Y(F(\text{James})) = 1946$, the year my father was born. And $F(Y(\text{James}))$ makes no sense as the year I was born has no biological father.

Comment: The notation we use for functions can be confusing here. If we write $h(a(w))$, for instance, then we must assume that w is a general symbol for an input and that h and a are the names of functions. The order in which these functions were applied is reverse to what many expect: first a was applied to w and then h was applied to $a(w)$ (and we must also assume that w is a valid input for the function a).

Question: With the function machines f and g shown in figure 2, draw a picture of a linkage of machines that takes an input x and eventually produces $f(f(g(f(g(x)))))$.

Quick Answer: Draw a g machine on the left with output hose going into an f machine with output hose into a g machine with output hose going into an f machine with output hose going into another f machine.

Given a function $f: X \to Y$ and a function $g: Y \to Z$, the function that assigns an element x of X the element $g(f(x))$ of Z is called the *composition* of the functions f and g.

Just to make matters particularly confusing, some people like to denote the composition of f and g as $g \circ f$, which is to be read backwards: The function $g \circ f$ first applies f to an input and then applies g to the result.

People might also write $(g \circ f)(x) = g(f(x))$ or sometimes just $g \circ f(x) = g(f(x))$, which are each probably more confusing than helpful.

Question: Can you unravel what $(g \circ f)(x) = g(f(x))$ is actually saying?

Quick Answer: It says: "The function with the strange name $g \circ f$ assigns to each input x the output $g(f(x))$."

If F is the father function and Y the year of birth function described earlier, then $Y \circ F$ is meaningful (it is the function that assigns to each person the year his or her biological father was born). Also $Y \circ F \circ F$ is meaningful. (It is the function that assigns to each person the year his or her paternal grandfather was born.) The composition $F \circ Y$ is not meaningful.

Sometimes it is helpful to read the little "o" for the composition of functions out loud also as "of." For instance, the function $Y \circ F$ is the "year of birth of the father of … " function, and $Y \circ F \circ F$ is the "year of birth of the father of the father of …" function, and so on. Reading $F \circ Y$ out loud ("the father of the year of birth of …") makes it clear that this composition is meaningless.

Comment: Even though using the word "of" forces us to read $g \circ f$ from left to right, we must remember that it is the rightmost function f that is applied first to a given input x and then the function g. This mismatch of direction is a result of our initial choice of notation for a function output. Life would be considerably easier if we wrote $(x)f$, or even just $x\ f$, for the result of applying the function f to an input x. Then

$x\ f$ reads:
start with x and then apply f to it.
$x\ f\ g$ reads:
start with x and then apply f to it and then apply g to the result.

and so on. All would be consistently left to right.

If x represents me, James, and f the father function, then the $f(x)$ notation follows the language "the father of James." The $x\ f$ notation follows the language "James' father." The mathematics community has settled on the first style.

Comment: In early grades students are taught to write the product of two numbers three ways. For example, two times three could be written with a cross symbol, 2×3, or with a raised dot, $2\cdot 3$, or with parentheses, $2(3)$. This third way matches function notation, which adds yet another layer of possible confusion. (Many students, when they first see the notion $f(x)$ naturally think $f\times x$.) It would be deliciously confusing if I called the function that adds two to each and every number "2". In this context, we'd have $2(3)=5$. As is always the case in reading mathematics context is important.

Question: Let $M:\text{People}\to\text{People}$ be the function that assigns to each person his or her biological mother, $F:\text{People}\to\text{People}$ the function that assigns to each person his or her biological father, $A:\text{People}\to\text{Whole Numbers}$ the function that assigns to each person his or her age in years, and $L:\text{People}\to\text{Letters}$ the function that assigns to each person the first letter of his or her name.

Which of the following compositions of functions are meaningful? For those that make sense, describe what the composite function is doing.

a) $L\circ M\circ F\circ F$
b) $A\circ L\circ L$
c) $L\circ A\circ F$
d) $M\circ L\circ F$

Quick Answer: Only the first is meaningful. It gives the first letter of a person's particular great grandparent's name.

ITERATED FUNCTIONS

One can sometimes compose a function with itself to obtain an *iterated* function.

For example, if M is the function that assigns to each person his or her biological mother, then $M\circ M$ assigns to each person his or her maternal grandmother, $M\circ M\circ M$ his or her maternal grandmother, and so on.

If

$$S:\text{Counting Numbers}\to\text{Counting Numbers}$$

is the successor function, it assigns to each counting number a the next counting number $a+1$, then

$$S(a)=a+1$$
$$S(S(a))=a+2$$
$$S(S(S(a)))=a+3.$$

To iterate a function, each output of the function must be a valid input for the function.

Writing lists of compositions and writing lists of nested parentheses soon becomes tiresome. If f is a function that can be iterated, then, for a counting number k we write f^k for that function composed with itself k times.

$$f^k(x) = \overbrace{f \circ f \circ \cdots \circ f}^{k \text{ times}}(x) = f\left(f\left(\cdots f(x)\right)\right)$$

.

We have the function machine picture of Figure 3.

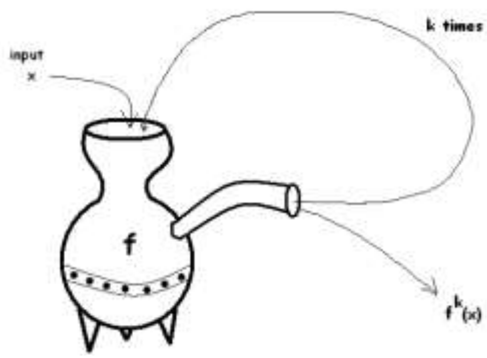

Figure 3.

For example, $M^2(\text{James})$ is James' maternal grandmother and $S^n(a) = a + n$.

Question: If a and b are counting numbers, explain why $f^a \circ f^b = f^{a+b}$.

Quick Answer: Applying a function f to an input first b times and then a more times has the same effect as applying the function f to that input $a+b$ times.

Comment: Annoyingly, this composition notation for iteration is abandoned when it comes to trigonometric functions. For example, $\sin^2(x)$ is taken to mean $(\sin(x))^2 = \sin(x) \times \sin(x)$, and not the composition $\sin(\sin(x))$. Annoying indeed!

GOING QUIRKY

Assume $f : X \to X$ is a function that can be iterated.

Suppose we run an input x through the function machine f zero times, that is, we do nothing with the input. Then we are left holding the same value x. For this reason, it seems appropriate to declare

$$f^0(x) = x,$$

that is, to declare f^0 to be the *identity* function (assign to each input itself).

Going further …

Can we give meaning to f^{-1}? Does it make sense to take a value x and run it through the machine -1 times?

Perhaps this means we are attempting to run a value x backwards through the machine.

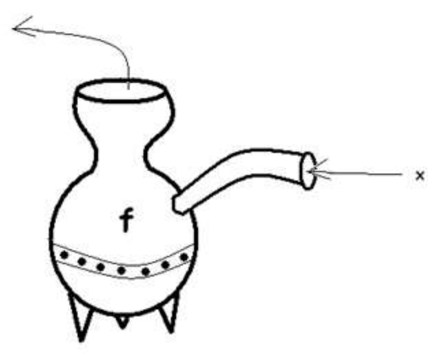

Figure 4.

To make sense of this, we must first assume that the value of x we are working with is in the range of f. Then, in thinking of the backwards operation of the machine, we see that we are looking for an input a whose output is x. Then $f^{-1}(x)$ would be a. That is,

$f^{-1}(x)$ is the input that gives the output x.

This, of course, is assuming that such an input exists and is clearly defined. Depending on the function, this might or might not be the case.

Example: Consider the function $R : \text{Counting Numbers} \to \text{Counting Numbers}$ which assigns to each counting the number the number with its digits reversed. For example, $R(123) = 321$ and $R(8) = 8$.

Each counting number is an output of this function and each output comes from a unique input. Thus R^{-1} is meaningful, and we have, for instance, $R^{-1}(321) = 123$ and $R^{-1}(8) = 8$. (In fact, R^{-1} is identical to the function R.)

Example: Consider the function $F : \text{Men who are fathers} \to \text{Men who are fathers}$ which assigns to each father his biological father. Then F^{-1} attempts to assign to each father his son. This is problematic as not all fathers have a son and, of those who do, some have more than one son and it is not clear which son to assign.

Question: Can this example be salvaged to some degree?. Consider

$$F : \text{Males} \to \text{Men who have at least one son}$$

which assigns to each male his biological father. Set

$$S : \text{Men who have at least one son} \to \text{Males}$$

to be the function that assigns to all fathers of at least one son his oldest son. (We have a mismatch of domain and range now.) Is $F \circ S$ the identity function? Is $S \circ F$ the identity function?

Quick Answer: The first is the identity function, the second is not.

If $f : X \to X$ is a function such that for each x in X there is a unique element a in X which has output x (see figure 5), then f^{-1} is defined to be the function that assigns to x the input whence it came: $f^{-1}(x) = a$. The function f^{-1} is called the *inverse* function to f.

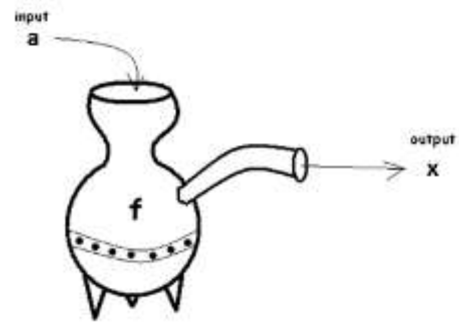

Figure 5.

Question: Suppose $f : X \to X$ has an inverse function. Explain why $f\left(f^{-1}(x)\right) = x$ and why $f^{-1}\left(f(x)\right) = x$.

Quick Answer: By definition, $f^{-1}(x)$ is the element that gives x as an output. So if we apply f to $f^{-1}(x)$ we get the output … x! That is, $f\left(f^{-1}(x)\right) = x$.

By definition, $f^{-1}\left(f(x)\right)$ is the input that gives the output $f(x)$. Clearly x has output $f(x)$, and so $f^{-1}\left(f(x)\right)$ is x.

(The reasoning here is tight and tautological. It is hard to wrap your brain around it.)

Assuming we can run a value x backwards through a machine twice, it is reasonable to denote the final result as $f^{-2}(x)$. This is the composite function $f^{-1} \circ f^{-1}$ applied to x. In general, f^{-k} is the function f^{-1} composed with itself k times.

One can reason that $f^{a} \circ f^{b} = f^{a+b}$ holds even if a and b are integers.

Question: Can one give meaning to $f^{\frac{1}{2}}$? Presumably, this is a function with the property that $f^{\frac{1}{2}} \circ f^{\frac{1}{2}} = f$?

Quick Answer: This is a surprisingly subtle question. See the essay "Compositional Square Roots" for some thoughts on this. http://tinyurl.com/zhkthfn.

Comment: Having just learned that the standard notation for iteration is abandoned when it comes to trigonometric functions ($\cos^3(x)$, for instance, means the value $\cos(x)$ cubed), the notation for inverse functions is *not* abandoned: $\sin^{-1}(x)$ does represent the inverse operations to the sine function, finding an input (angle) whose output is x.

To summarize: If k is a positive integer, then

> $\sin^k(x)$ follows the algebra interpretation (it means $\sin(x)$ raised to the kth power) and not the iteration interpretation,

but

> $\sin^{-1}(x)$ follows the iteration interpretation (it means the inverse to sine) and not the algebra interpretation (which would be $\frac{1}{\sin(x)}$).

Very confusing!

By the way, I personally do not know what $\sin^{-2}(x)$ would mean.

MULTI-VALUED FUNCTIONS

Consider the operation $Sqrt$ that assigns to each square number $1, 4, 9, 16, \ldots$ its square roots. So for the number 9 we assign the set $\{-3, 3\}$.

Some might argue that this operation is not a function since for each input we are assigning more than one output. But we can view this operation as a function. We have

$$Sqrt : \text{Square Numbers} \to \text{Sets of Integers}$$

with the rule: assign to each square number its set of square roots.

A multi-valued function is a function: it's just one whose outputs are sets.

Some textbooks make a distinction between a function and a "relation," with any operation that might assign more than one output to a given input being called a relation. But even these relations can be viewed as functions and there is no need for fussing.

Comment: For those who like formal thinking … A *function* from a set X to a set Y is a subset $f \subseteq X \times Y$ with the property that each x in X appears as a first element of precisely one pair in f. A *relation* from a set X to a set Y is subset $R \subseteq X \times Y$ with the property that each x in X appears as a first element of at least one pair in R.

SOME SWIFT COMMENTS ON NUMERICAL FUNCTIONS

School mathematics places a strong focus numerical functions with domain the set of real numbers (or just a subset of the reals) and range within the set of real numbers too: $f:\mathbb{R}\to\mathbb{R}$.

Here it is almost universally agreed to denote a general input of such a function with the symbol x and a general output with the symbol y. In writing

$$y=f(x)$$

we are making a claim that y equals the output $f(x)$ for a given input x. The claim might or might not be true.

Example: Suppose f is the squaring function: it assigns to each real number x its square x^2. Then $25=f(5)$ is a true statement and "$-3=f(x)$ for some x" is a false one.

It often happens that we can describe the rule of a numerical function by a formula and here people often muddle the meaning of an equation and the definition of a function.

An Aside on Equations

Mathematics is a language. For those reading this essay, that language at present is English. For example, the statement "$2+4=6$" has a noun (the quantity two-plus-four), a verb (equals), and an object (six). The statement

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

is the sentence "One half plus one third is equivalent to three sixths plus two sixths, which is five sixths."

Some mathematical sentences are true ($2+4=6$, for instance) and some are false ($1=2$, for example). Mathematics is usually interested in sentences that are true.

But one usually needs a context to determine the truth value of a statement. For example, we cannot assess the truth of the claim "Suzy is less than five feet tall" without knowing which particular Suzy of the world we are referring to.

The same is true of the mathematical statements involving variables, such as "$x^2=25$." Some values of x make this a true statement, others do not. In, and of, itself, the statement $x^2=25$ has no truth value. (But we can't help but think of the numbers 5 and -5, the values that make this statement a true one about numbers. We are compelled to seek truth!)

In mathematics, an equation is any statement that claims that two expressions are equal. If variables are involved, we cannot say whether or not the equation speaks truth. The work of an algebra class is to find which values of the variable(s) involved make the given equation a true statement about numbers.

Defining Numerical Functions

One will often read in textbook a statement such as

$$f(x)=1+\sqrt{x}.$$

This is an equation. It says that "The output of a function given by a function whose name is f is $1+\sqrt{x}$ if x is the input to that function." Without knowing any details of the function f we have no way to assess whether or not this is a true or false statement, or which particular values of x make it true.

However, authors will write an equation like this not as an equation to be studied, but as way of defining a function whose name is f.

Example: Without being told otherwise, we would likely presume the statement

$$h(x) = \frac{x^2 + x^3}{2}$$

is to interpreted as: "A new function $h : \mathbb{R} \to \mathbb{R}$ is being considered. It takes an input x and assigns to it the average of its square and cube, $\frac{x^2 + x^3}{2}$."

Comment: This abuse of mathematical notation can be confusing. Many mathematicians prefer to write

$$\begin{aligned} h : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{x^2 + x^3}{2} \end{aligned}$$

with an ordinary arrow to show between which two sets a function operates and a barred arrow to show how a generic input is transformed. This notational approach is much clearer, but it is rarely used in school mathematics.

The muddled notation actually goes a little deeper still. An equation such as $y = 2x + 3$, for instance, is just an equation. Some particular values for x and y make it a true statement about numbers, other pairs of values do not.

But because we have become so ingrained to think of x as an input and y as an output, many people, without thinking, will associate with this equation a function: the function from $\mathbb{R}$ to $\mathbb{R}$ which takes a generic input x and associates to it the output $2x + 3$. Notice no name for this function is indicated.

In summary, here is the muddled situation. Even though " $f(x) =$ some expression involving x" is an equation that might or might not be true depending on the value of x and what the function f does, such a statement is often interpreted as a definition of a function named f.

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \text{The value of that expression} \\ &\quad\;\; \text{for that value of } x \end{aligned}$$

Even though " $y =$ some expression involving x" is an equation that might or might not be true depending on the chosen values for x and y, such a statement is often interpreted as the definition of an unnamed function from $\mathbb{R}$ to $\mathbb{R}$ that takes an input x and assigns to it the value of that expression for that value of x.

If a function is defined solely with a formula, then it is assumed that the domain of the function is the subset of all real values for which that formula makes sense.

Example: If we are told simply that a function a is given by $a(x) = \frac{1}{x}$, then we should probably assume its domain is the set of all real numbers different from zero.

We can write:

Domain = $\{x \mid x \neq 0\}$ or $\{x \in \mathbb{R} \mid x \neq 0\}$.

This is read as "the set of all real numbers x such that x is different from zero."

It is possible that we might be later told that we are to consider this function only on positive inputs, for example. In which case the domain of the function is just the set of all positive real numbers.

Example: If $w(x) = \sqrt{x-1}$, then we should assume that the domain of this function is the set of all real values greater than or equal to one.

Domain = $\{x \mid x \geq 1\}$.

Comment: There are multiple ways to express regions of the real number line. For example, the set of all non-negative real numbers might be described simply that way, in words (there is no problem with writing words in mathematics) or in set notation as $\{x \mid x \geq 0\}$ or in interval notation $[0,\infty)$, or by shading a number line with a closed dot to represent the inclusion of that value (and an open dot to indicate the exclusion of that value):

.

Question: What is the largest possible domain of f given by $f(x) = 2 + \sqrt{x+4}$, and for that domain, what is the range of the function?

Quick Answer: The given expression defining the function is meaningful only for $x \geq -4$. The largest possible domain is

Doman = set of all real numbers greater than or equal to -4

$= \{x \mid x \geq -4\}$

$= (-4,\infty)$.

(Any one of these descriptions of the domain is acceptable.)

What possible outputs can appear? Each output equals two plus a non-negative quantity. (And we see that the input $x = -4$, gives the output 2 itself.) Every value 2 or greater does appear as an output.

Range = set of all real values greater than or equal to 2

$= \{y \mid y \geq 2\}$

$= [2,\infty)$

Comment: It is common practice to use the symbol y for outputs. This is just a social convention. There is no mathematical reason to follow it.

A VISUAL REPRESENTATION OF NUMERICAL FUNCTIONS

Numerical functions are extra pleasing as we can often provide beautiful visual representations of them. These representations are called *graphs*.

I gave an indication of my approach to this matter in my October 2015 Curriculum Essay http://tinyurl.com/jvaueel .

If this essay has the feel of, say, an overview guide for students and teachers, you are right! I am in the midst of writing *FUNCTIONS AND GRAPHS: A Clever Study* Guide, a third book in a series for the MAA. I have just shared material that appears in chapters 1 and 3. (This is also material adapted from my THINKING MATHEMATICS! series.)

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