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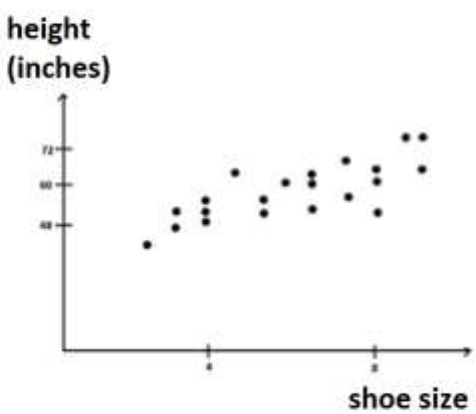
TANTON'S TAKE ON …

GRAPHING

OCTOBER 2015

In my experience middle- and high-school students have little trouble plotting and reading data in a scatter plot.



I know it takes some work to help young students develop a level of comfort with such plots, but once matters "click" they seem to click relatively well for all.

So why not bounce off of this comfort and have students view all acts of graphing in middle- and high-school mathematics as a data plotting?

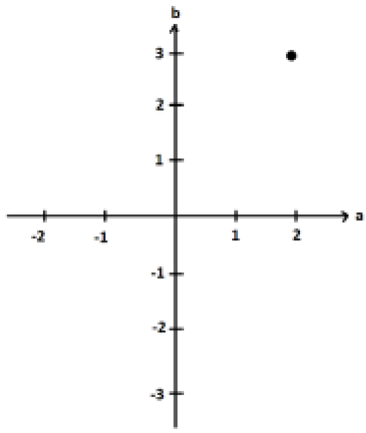
DATA FROM EQUATIONS

Every equation provides an opportunity to gather data. For example, here's an equation in two variables, a and b:

$$b^2 = a^3 + 1.$$

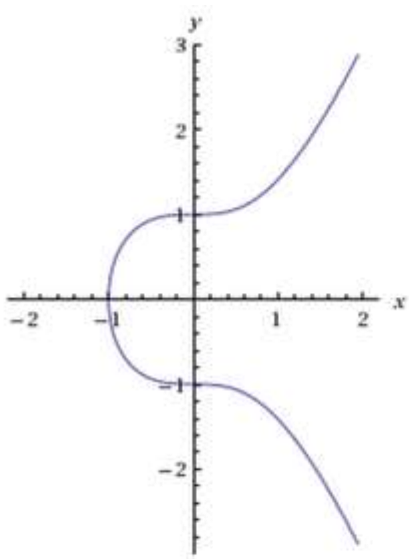
The natural data to collect from it is the set of values for a and b that make the equation a meaningful and true statement about numbers. For example, $a = 2$ and $b = 3$ gives a true number statement:

$(3)^2 = (2)^3 + 1$. We have one data value to put on our scatter plot.



We also have $a = 2$, $b = -3$ and $a = 0$, $b = 1$, and $a = 0$, $b = -1$, and $a = -1$, $b = 0$ as data points. Are there any more? (Tough Question: Are there any more data values with integer entries?)

We certainly have many more non-integer data points to plot. For example, $a = 4$, $b = \sqrt{65} \approx 8.1$, and $a = \sqrt[3]{24} \approx 2.9$, $b = -5$ both work. A deep interesting discussion reveals that there is a whole continuum of data values to plot, giving a scatter plot that looks more like a curve than a plot of individual points.

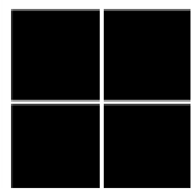


Definition: The <u>graph of an equation</u> is a scatter plot of all the data values that make the equation a meaningful and true number statement.

One can have some quirky fun with this idea.

EXAMPLE: *Consider the equation:* $\frac{xy}{xy} = 1$. *Describe its associated graph.*

Answer: The equation $\frac{xy}{xy} = 1$ is meaningful and true whenever x and y each have a non-zero value. The equation is not well-defined if one, or both, of these values is zero. Thus the graph of this equation is the set of all points in the coordinate plane away from the axes.



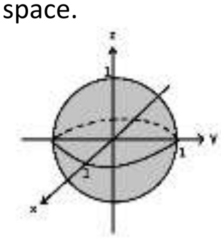
EXERCISE: *What is the graph of the equation* $\frac{\sqrt{x}\cdot\sqrt{y}}{\sqrt{x}\cdot\sqrt{y}} = 1$*?*

EXERCISE *What is the graph of* $\left(\frac{2x}{|x|-\mathrm{x}} + 1\right)\left(\frac{2y}{|y|+\mathrm{y}} - 1\right) = 0$ *?*

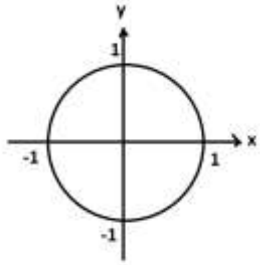
EXERCISE: *What is the graph of a mathematical identity?*

Graphs aren't always two-dimensional.

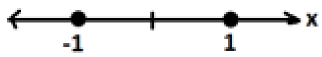
EXAMPLE: The graph of the equation $x^2 + y^2 + z^2 = 1$ is a sphere in three-dimensional space.



The graph of the equation $x^2 + y^2 = 1$ is a circle in two-dimensional space.



The graph of the equation $x^2 = 1$ is a circle in one-dimensional space.



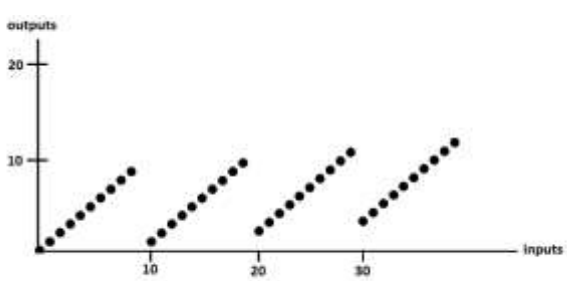
Comment: Context, as always, is important! If $x^2 + y^2 = 1$ is actually an equation in three variables x, y, and z, but the variable z just doesn't happen to be mentioned (maybe the equation is $x^2 + y^2 + 0 \cdot z = 1$), then the graph of $x^2 + y^2 = 1$ is a cylinder sitting in three-dimensional space.

EXERCISE: *What is the graph of $x^2 = 1$ as an equation in two variables x and y? In three variables x, y, and z?*

DATA FROM FUNCTIONS

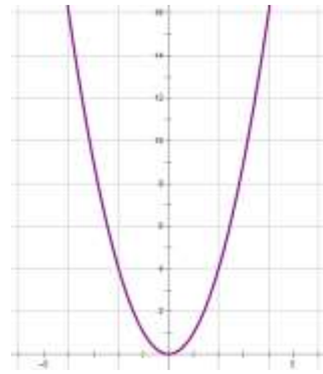
The natural data to collect from a numerical function F is the set of allowable inputs for the function, each paired with its matching output.

For example, the function that assigns to each counting number the sum of its digits gives a scatter plot that looks like this.



Question: *How does this plot appear in the 100s? In the 1000s?*

If F is the squaring function (each real number as input is matched with its square as output) we get the familiar parabolic curve.



Definition: The <u>graph of a numerical function</u> F is the scatter plot of all the data values (p, q) with p a valid input of the function and q its matching output.

Question: *If (p, q) is a data point for the function F, then (q, p) is a data point for the inverse function F^{-1} (if the inverse is meaningful). How does the graph of F^{-1} thus appear? (Must F^{-1} exist to play with this idea?)*

SUBTLE PEDAGOGICAL COMMENT:

All schoolbook functions seem to be described by formulas. For example, the squaring function F might simply be described as $F(x) = x^2$. This is expression is to be interpreted as "F is the function that associates to a real number input x the output given by evaluating x^2." This is fine: it is just a wordless description of the rule F follows.

But matters are delicate because, in the same stroke of chalk, people might then write:

$$y = F(x).$$

This is now an equation. And this equation might be true for specific values of x and y, or it might not be true. In fact, it is only true if a given real number x happens to be a valid input for the function F and the given real number y is its matching output.

Thus a data point (x, y) is a valid data point for the equation $y = F(x)$ precisely when (x, y) is a valid data point for the function F. The graph of the equation $y = F(x)$ is thus identical to the graph of the function F.

To reiterate… Writing $y = x^3 + 1$, for example, doesn't, in and of itself, define a function. This is an equation and this equation certainly has a graph.

However, we educators are so used to writing "$y = F(x)$" that we naturally think an equation like this as defining a function. (Take F to be the function that associates to a real number x the output $x^3 + 1$.) We even say that the graph of the equation is the graph of a function without thinking through the two layers at play here. But there are two layers.

It is worth pointing this out carefully (but not in a labored way!) to students. Thinking of graphs as scatter plots of data helps with this.

A PEDAGOGICAL DISADVANTAGE

The scatter plot approach suffers the same problem as the traditional approach in that it is a static viewpoint and doesn't encourage students to think dynamically: "As one quantity (variable) changes value this way, the second quantity (variable) responds by changing this other way."

Thinking through variation and covariation should also be encouraged very early on, and steadily, throughout the curriculum. Like science class, it provides another opportunity to collect and plot data. Then practice is needed for the reverse: from a graph, describe the variation and covariation it represents for the two variables at hand. (After all, variables are called *variables*.) We need to encourage the variation thinking too.

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