CURRICULUM INSPIRATIONS: www.maa.org/ci

MATH FOR AMERICA_DC: www.mathforamerica.org/dc

INNOVATIVE CURRICULUM ONLINE EXPERIENCES: www.gdaymath.com
TANTON TIDBITS: www.jamestanton.com

TANTON'S TAKE ON …

POLYNOMIAL DIVISION

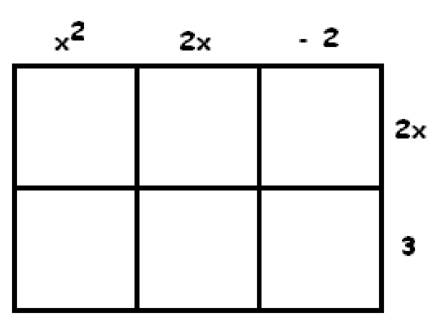
NOVEMBER 2014

The area model provides a wonderful means for conducting polynomial multiplication and, more fun, polynomial division. It sure beats the traditional long-division algorithm, which is tedious and unenlightening. (I still love the exploding dots approach for its conceptual ease – see lesson 1.6 and 1.7 of www.gdaymath.com/courses/exploding-dots - but the area model approach is much easier to conduct, in practice, on paper.)

The material that follows appears in
THINKING MATHEMATICS! Vol 1: Arithmetic = Gateway to All
available at www.lulu.com.

Here's how to multiply x^2+2x-2 and $2x+3$, for instance.

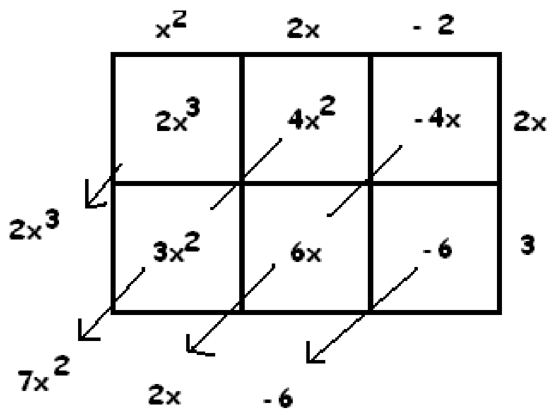
Begin by drawing a rectangle divided into as many columns as there are terms the first polynomial and as many rows as the count of terms of the second polynomial.



To compute the product of the polynomials simply multiply cell by cell and add the results.

(This geometric display of the arithmetic matches precisely the algebra of the distributive rule. It is delightful that this array also aligns with the geometry of area – even with the quirky feature of potentially having negative side-lengths!)

Adding along the diagonals has the convenience of automatically collecting like powers of x.



The answer $2x^3 + 7x^2 + 2x - 6$ appears.

EXERCISE: Compute the following products this way. (WARNING: Be sure to draw rows and columns for appropriate zero terms!)

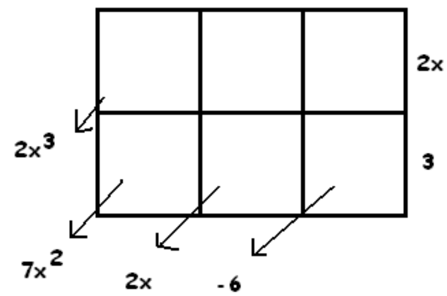
a) $\left(3x^5 + 2x^4 - 5x^3 + 4x^2 - x + 10\right)\left(x^2 - 3x + 4\right)$

b) $\left(x^3 - 3x + 2\right)\left(2x + 5\right)$

c) $\left(x^3 - x\right)\left(x^3 + x^2 - x - 1\right)$

A nice feature of the "area method" is that we can execute it backwards: If given the answer and one of the original polynomials, it is possible to logically deduce what the entries of the table must be, as well as the identity of the missing polynomial.

EXERCISE: Here is the same table as before with the first polynomial missing and the entries of the table blank.



a) What must the top left cell of the table be?
b) What must be the first term of the missing polynomial?
c) What must the remaining entry of the first column be?
d) Continue in this way to fill in the table and to show that the missing polynomial is indeed $x^2 + 2x - 2$.

The above example has, in effect, computed

$$\frac{2x^3 + 7x^2 + 2x - 6}{2x + 3}.$$

EXERCISE: Solve the following division problems using the reverse area method. (To do this, one must first determine what size grid to draw. The number of rows is always clear. The number of columns requires a little thought.)

a) $\dfrac{2x^2 + 7x + 6}{x + 2}$

b) $\dfrac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$

c) $\dfrac{x^6 - x^5 + x^4 - 2x^3 + 6x^2 - 6x + 4}{x^4 - 2x + 4}$

d) $\dfrac{x^8 - 1}{x + 1}$. (Is $1028^8 - 1$ prime?)

COMMENT: One can deduce the number of columns one needs in a division problem by taking note of the highest powers of x that appear in the problem. Note, for example, multiplying $4x^4+2x^3-x^2-x+3$ and x^2-2x+1 together produces a polynomial with highest power x^6. On the other hand, dividing $4x^6-6x^5-x^4+3x^3+4x^2-7x+3$ by x^2-2x+1 must yield an answer with highest power x^4. (Its answer times x^2-2x+1 must produce $4x^6-6x^5-x^4+3x^3+4x^2-7x+3$.) Thus we can deduce that the division problem:

$$\frac{4x^6-6x^5-x^4+3x^3+4x^2-7x+3}{x^2-2x+1}$$

requires a table with five columns, one for each of the powers 1, x, x^2, x^3, and x^4.

EXERCISE: Compute $\frac{4x^6-6x^5-x^4+3x^3+4x^2-7x+3}{x^2-2x+1}$, but use a table with three rows and EIGHT columns. Verify that misjudging the number of columns to the excess offers no cause for concern!

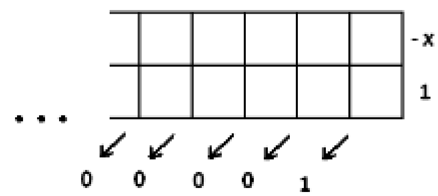
EXERCISE: Algebra students are often taught to memorize the difference of two squares formula, $x^2-a^2=(x-a)(x+a)$, and the difference of two cubes formula $x^3-a^3=(x-a)(x^2+ax+a^2)$,

Compute $\frac{x^2-a^2}{x-a}$ and $\frac{x^3-a^3}{x-a}$ via the reverse area method and verify these formulas.

Here is a curious way to go further:

EXERCISE:

a) Complete the following table, with infinitely many columns to the left, to evaluate $\frac{1}{1-x}$.



b) Compute $\frac{1}{1+x}$ via the reverse area method.

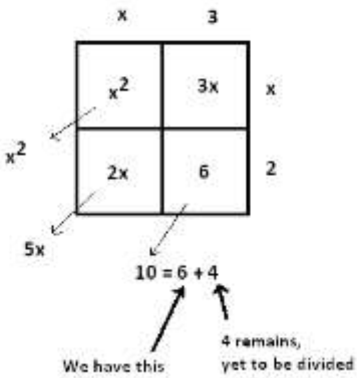
c) Compute $\frac{1}{1-x^2}$ via the reverse area method.

d) Compute $\frac{1}{1-x-x^2}$ via the reverse area method. What famous sequence of numbers do you see appearing?

REMAINDERS

The reverse area method shows that $\frac{x^2+5x+6}{x+2}$ equals $x+3$. (Check this!)

Consequently, dividing $x^2+5x+10$ by $x+2$ must leave a remainder of 4. Let's see how this appears in the table as we work through the problem:



The table does indeed show:

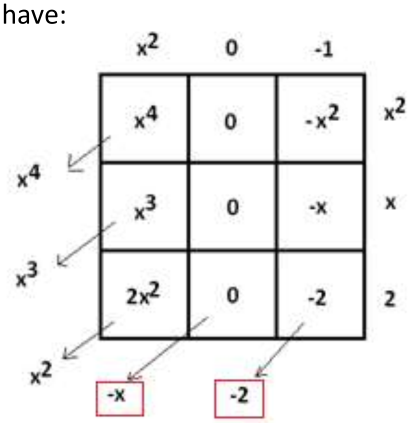
$$\frac{x^2+5x+10}{x+2} = x+3+\frac{4}{x+2}.$$

(Multiply through by $x+2$ to check this.)

As another example, consider:

$$\frac{x^4+x^3+x^2+x+1}{x^2+x+2}.$$

We have:



This reads:

$$\frac{x^4+x^3+x^2-x-2}{x^2+x+2} = x^2-1$$

which is not what we wanted. But by focusing on what we do want, we deduce:

$$\frac{x^4+x^3+x^2+x+1}{x^2+x+2} = \frac{x^4+x^3+x^2-x-2 \quad + \quad 2x+3}{x^2+x+2} = x^2-1+\frac{2x+3}{x^2+x+2}$$

The remainder of $2x+3$ is now clear.

EXERCISE: Compute $\frac{x^5-x^3+x+1}{x^4-2}$.

EXERCISE:

a) Use the reverse area method to evaluate $\frac{x^3+8}{x+2}$.

b) Making this a little more abstract, evaluate $\frac{x^3+a^3}{x+a}$.

c) Use the reverse galley method to show that x^5+a^5 is divisible by $x+a$.

d) Explain why, for n odd, x^n+a^n is always multiple of $x+a$.

e) Show that $7^{50}+22^{25}$ is divisible by 71.

(People forget in algebra class that x can actually be a number!)

tanton.math@gmail.com