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TANTON'S TAKE ON …

FACING NUANCE AND AMBIGUITY

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Mathematics is often seen as the epitome of a study of hard facts: answers are either right or wrong, concrete algorithms give concrete results, and mathematical claims are either true or false. All seems somewhat binary with no place for ambiguity or nuance.

Of course, there are fundamental questions mathematicians still do not know the answer to. Are there infinitely many examples of "twin primes," that is, pairs of consecutive odd numbers like 11 & 13, and 41 & 43 both of which are prime? Do the equations governing fluid flow actually have smooth mathematical solutions? In the world of research mathematics, the unknown, partial answers, and improvisation are par for the course.

But we don't teach the confidence to pursue the unknown in school mathematics and usually give the impression that all is known in any case.

Yet students and educators do contend with ambiguity and uncertainty, regularly so when studying and thinking about the school curriculum. They just rarely speak about it.

Because of the societal view that math is concrete, having questions or doubts about what one is studying is often interpreted as a sign of inadequacy, if not failure. Yet I would say that the students and educators having those doubts are deep thinkers and are engaging in effective learning. They should be encouraged to explore the haze, not run from it just to catch up with the textbook curriculum that blithely carries on. Thinking deeply about allegedly simple ideas is the path to extraordinary understanding and impact.

Let me give an example of a solid math topic that, upon reflection, we see is deeply nuanced and potentially very confusing, despite its algorithmic simplicity: grade-school division.

Consider the division problem: $20 \div 4$. What does this mean? What are we being asked to compute?

I see five approaches taught in a typical school curriculum.

Division as finding groups: $20 \div 4$ is asking for how many groups of four can be found in a collection of twenty.

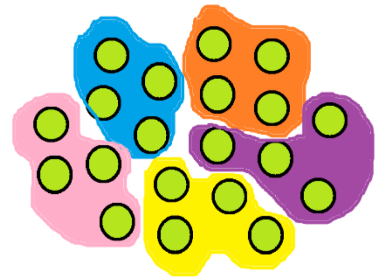
Division as sharing: $20 \div 4$ is asking for how many pies each person receives if twenty pies distributed equally among four people.

Division as reverse multiplication: $20 \div 4$ is asking: What times four gives the answer twenty?"

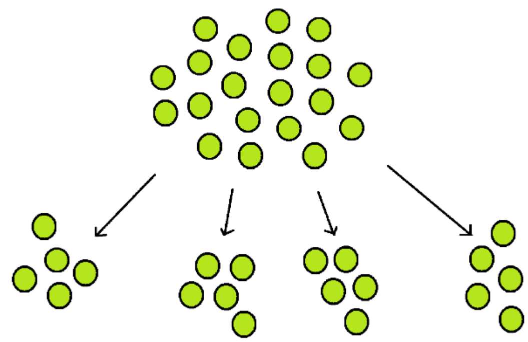
Division as repeated subtraction: $20 \div 4$ is asking for the number of times I need to subtract four from twenty to get to zero.

Division as multiplication of fractions: $20 \div 4$ is really $\frac{1}{4} \times 20$.

These models of division at are typically introduced at different times throughout a curriculum, often simply when it seems convenient to switch gears to analyze a new type of problem. Sometimes there is little or no justification as to why a change of perspective is valid. Is it philosophically obvious that the processes of group-finding and of sharing, for example, give the same numerical outcomes each and every time?



DIVISION AS FINDING GROUPS
There are FIVE groups of 4 in a picture of 20



DIVISION AS SHARING
Distributing 20 items equally among 4 people gives FIVE objects per person

This is a lovely matter to ponder upon! And beautiful mathematics instruction does dwell on these matters and invites learners to wonder, ponder, and struggle with such issues.

In my experience, students typically tend to focus on the "division as grouping" model as the basis idea for their thinking and then work to extend that model to make sense of the other four. For instance, look at the first picture of the various groups of four highlighted in a picture of twenty pies. These groups of four show how to share the pies:

> Identify one group of four and then give one pie from that group to each recipient. Identify a second group of four and give each recipient one more pie from it. And so on. As there are five groups of four in total, each recipient receives five pies.

> In general, in the sharing model, each recipient receives one pie for each group identified in the group model and so receives as many pies as there are groups.

The same picture also shows twenty as "five groups of four" and so explains the reverse multiplication model, and it also shows that we can subtract four from twenty five times to thus explain the repeated subtraction model.

But the fraction model …

In school mathematics, we have been trained to say that "of" means multiply. So when we read $\frac{1}{4}\times 20$, we immediately say "a quarter of twenty." And the sharing model picture on the shows that a quarter of twenty is five. This seems well and good. But there is something subtle and perturbing here. It is not clear that "$\frac{1}{4}$" and "20" are being used in the same way. We have a number, 20, and actual collection twenty pies, but we're not using $\frac{1}{4}$ as number here, more as an instruction: *find a quarter of …*. So fractions are not numbers, they are instructions?

Yet we are taught that fractions ARE numbers. They represent portions of pie.

$$\frac{1}{2} + \frac{1}{3} = \frac{7}{8}$$

And we can add and subtract portions of pie, so we can add and subtract fractions: they are numbers after all. We see, for instance, that $\frac{1}{2}+\frac{3}{8}$ gives almost a whole pie, $\frac{7}{8}$.

But if fractions are numbers, then we should be able to multiply them as well. And now I am stuck: What on earth does it mean to multiply portions of pie?

I am discovering now that there really is something deep and confusing about fractions. My understanding of "$\frac{1}{4}\times 20$" is thus in doubt and our fifth division model might not be clear and obvious after all.

Mathematics, especially "elementary" school mathematics, is chock full of nuance and doubt. Rather than side-step, if not run from, the haze and the confusion, we can view ambiguity and nuance as an invitation to probe and discuss and explore. We might not be able to fully answer all our questions and we might in the end simply identify a list of things the world just seems to believe about fractions (and not know why). But identifying what you don't know is a powerful enterprise, one that allows you to move forward nonetheless with clarity and intellectual conviction.

It is wonderful that mathematics classes too can provide the opportunity to teach the power—and delight—of identifying what you don't know as a basis for taking steps forward.

Post Script: Fractions actually are hard! See http://gdaymath.com/courses/fractions-are-hard/ for more.

GLOBALMATHPROJECT

Thank you all for helping make Global Math Week a world-wide phenomenon! Teachers purport reaching 1.77 million students from 168 different countries and territories. Just astounding and so very beautiful!

All was grassroots. All was volunteer. And all was propelled by a stunning community of teachers, coaches, leaders, enthusiasts – you! -- simply wanting to bring joyous mathematics to the world.

THANK YOU!

If you took part in #GMW2017 could you complete the exit survey here: http://www.surveygizmo.com/s3/3894595/GMW2017-Exit-Survey? We need to prove to educational institutions, with solid data, what we know to be true: that GMW was solid and real and had an impact! Your help with this, by completing the survey, would be valued tremendously!

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