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TANTON'S TAKE ON …

TWO DATA POINTS

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Last time I checked my savings account I had a balance of $\$275$. Checking just now I see that my balance has since grown to $\$825$. Woohoo!

There are two ways to think about this growth. Additive thinking has me say that I gained $\$550$ over this recent period of time: $825 = 275 + 550$. Multiplicative thinking has me say that I tripled my money: $825 = 275 \times 3$.

Let's call the time period between the two times I checked my balance one unit of time. If I believe these growth rates are constant, then additive thinking predicts I will have $275 + 550 + 550 = \$1375$ after another unit of time, and multiplicative thinking predicts a balance of $275 \times 3 \times 3 = \$2475$. (Let's hope it is multiplicative growth!) In general, after t units of time, additive thinking predicts a balance of $A(t) = 275 + 550t$ dollars and multiplicative thinking a balance of $M(t) = 275 \times 3^t$ dollars.

We have just found two equations that "fit" the two data points $(0, 275)$ and $(1, 825)$. The first equation is a *linear model* and the second an *exponential model*. It is a standard part of the school curriculum to have students find linear and exponential equations that fit two given data points.

But what is the best way to do this work if the two data points are not as "friendly"?

How might one go about finding a linear equation $a+bt$, and then an exponential equation $a\cdot b^t$, that fits the data $(3,201)$ and $(17,640)$?

t	a + bt
3	201
17	640

t	a·b^t
3	201
17	640

EXPONENTIAL FIT

Let's do the supposedly harder one first, the hard way.

The Brute Force Approach
We seek an exponential equation

$$M(t)=a\cdot b^t$$

with $M(3)=201$ and $M(17)=640$. We have two equations to work with.

$$201=a\cdot b^3$$
$$640=a\cdot b^{17}$$

Dividing the equations gives $b^{14}=\frac{640}{201}$ and so $b=\left(\frac{640}{201}\right)^{\frac{1}{14}}$. Now substitute this value into the first equation to see

$$a=\frac{201}{b^3}=201\left(\frac{201}{640}\right)^{\frac{3}{14}}.$$

Thus

$$M(t)=201\left(\frac{201}{640}\right)^{\frac{3}{14}}\left(\frac{640}{201}\right)^{\frac{t}{14}}$$

is a scary-looking exponential formula that does the trick. (We can simplify it a tad, I suppose.)

The "Avoid Hard Work" Approach
What makes this problem hard are the numbers. The growth is occurring over a period of time that starts at $t=3$ and ends at $t=17$, a span of fourteen units of time.

Life would be so much easier if the time period was one unit of time starting at $t=0$.

t	a·b^t
0	201
1	640

Although the growth rate is a bit unfriendly (the data has grown by a factor of $\frac{640}{201}$), it is conceptually straightforward to write an exponential equation that fits this data. The following works.

$$N(t)=201\left(\frac{640}{201}\right)^t.$$

Check: Put $t=0$ and $t=1$ into this formula.

But the real data isn't growing over a period of one unit of time: it grows over fourteen units of time. So we need to "slow this formula down" by a factor of fourteen.

t	a·b^t
0	201
14	640

How can we do this? How can we make fourteen units of time behave, in some sense, like one unit of time? How do we make $t=0$ and $t=14$ "behave like" $t=0$ and $t=1$?

Some mulling and toying suggests to replace t by $\frac{t}{14}$. So let's set

$$W(t) = 201\left(\frac{640}{201}\right)^{\frac{t}{14}}.$$

Check: Put $t = 0$ and $t = 14$ into this formula.

But the data we were given doesn't start at $t = 0$. It follows a span of fourteen units of time starting at $t = 3$.

t	$a \cdot b^t$
3	201
17	640

Can we adjust the expression we currently have so that $t = 3$ and $t = 17$ now behave like $t = 0$ and $t = 14$?

Some more mulling and toying suggests replacing t by $t - 3$. So set

$$M(t) = 201\left(\frac{640}{201}\right)^{\frac{t-3}{14}}.$$

Check: Put $t = 3$ and $t = 17$ into this formula.

That does it. We have a lovely exponential formula that matches the two given data points.

Challenge: Check that our formula here agrees with the brute-force answer we obtained earlier.

Although this approach seems longer and more involved, it teaches mathematical practice: work hard to avoid hard work! Start with the simplest version of the problem and build up ideas from there.

(By the way, the required "mulling and toying" required here is precisely the thinking students have likely already explored in studying the transformation of graphs comparing plots of $y = f(x)$, $y = f(x-k)$, and $y = f(kx)$.)

Just Write Down the Answer!

Seeing the structure of the equations obtained by this approach provides the confidence to simply write down fitting formulae. Not a lick of scratch work. Whoa!

Consider, for example, $(87, 123)$ and $(1000, 14)$. Can you just see that

$$M(t) = 123\left(\frac{14}{123}\right)^{\frac{t-87}{913}}$$

fits? (This equation represents exponential decay.)

Exercise: Write down an exponential equation that fits the data $(10, 10)$ and $(20, 20)$.

Exercise: Write down an exponential equation that fits the data $(3, 46)$ and $(15, 46)$, if you can.

Exercise: Write down an exponential equation that fits the data $(8, 11)$ and $(18, 0)$, if you can.

LINEAR FIT

Let's repeat our "avoid hard work" approach with additive thinking. Let's write down a linear equation for each of these three data tables.

t	a + bt
0	201
1	640

t	a + bt
0	201
14	640

t	a + bt
3	201
17	640

For the first table we can use

$$B(t) = 201 + 439t.$$

Slowing down by a factor fourteen gives

$$C(t) = 201 + 439 \cdot \frac{t}{14}$$

for the second table of values.

Making $t = 3$ “behave like” $t = 0$ gives

$$A(t) = 201 + 439 \cdot \frac{t-3}{14},$$

which is a linear equation fitting the third data table.

Comment: The standard curriculum approach is to use the brute-force method, but with one piece of sophistication. A line connecting the two data points will have slope $\frac{640-201}{17-13} = \frac{439}{14}$, and so a linear equation that fits the data will have the form $A(t) = \frac{439}{14}t + b$. We can now substitute in one data point to solve for b.

Here’s a little exercise that brings ideas to together, and pushes matters a little bit further too.

Exercise: Consider the data

x	y
3	4.2
6	1.8

a) Write down an exponential equation $y = a \cdot b^x$ that fits this data.

b) Draw a table of x and $\log y$ values. Find a linear equation that fits the x and $\log y$ data.

c) Do your equations in parts a) and b) match?

d) Find an equation of the form $y = a \cdot x^b$ that fits the two original data values. (Apply a logarithm first?)

e) Is there an equation of the form $y = x^a + b$ that fits the original data? (This is a yes/no question.)

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