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# *TANTON'S TAKE ON …*

# AVERAGES and BALANCE POINTS

**MAY 2017**

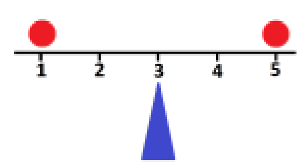
By the average of two numbers, we mean the number that lies halfway between them on the number line. For example, the average of $1$ and $5$ is $\frac{1+5}{2}=3$ and $3$ is equidistant from $1$ and $5$ on the line.

**Comment:** This definition of the average of two numbers $x$ and $y$, with $x \le y$, say, should be given by the formula

$$x+\frac{1}{2}(y-x).$$

This equivalent to $\frac{x+y}{2}$.

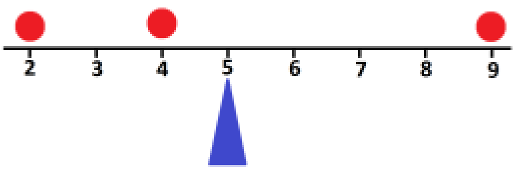
Physically, the average of two numbers is the location of the balance point of two $1$ gram marbles placed on a (weightless) number line.




Many school texts make the claim that the same is true for the arithmetic mean of more than two values:

*If $1$ gram marbles are placed at positions $x_1$, $x_2$, …, $x_n$ along a (weightless) number line, then the balance point of the system occurs at the arithmetic mean of the numbers:* $\frac{x_1+x_2+\cdots+x_n}{n}$.

For example, the arithmetic mean of $2$, $4$, and $9$ is $\frac{2+4+9}{3}=5$. Is it obvious that the corresponding system of marbles balances at position $5$?




Many texts point out that the total sum of distances of individual marbles to the left of the mean matches the total sum of distances of marbles from the mean to the right. In our example we have two marbles to the left of the mean, one $3$ units away the other $1$ unit away, and one marble to the right at a distance of $4$ units. And indeed $3+1$ equals $4$.

> **Challenge:** For a set of numbers $x_1$, $x_2$, …, $x_n$ with mean $m=\frac{x_1+x_2+\cdots+x_n}{n}$, prove that the sum
>
> $$(x_1-m)+(x_2-m)+\cdots+(x_n-m)$$
>
> is sure to equal zero. (The negative terms in this sum correspond to distances to the left, and the positive terms distances to the right.)

Does this observation about balanced left and right distances justify the claim that the mean matches the location of the balance point?

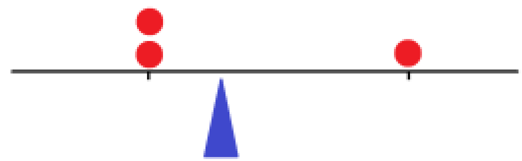
Many texts make this observation and end the conversation there thereby leading the reader think that all has now been fully clarified.

I personally don't think it is at all obvious that the balance point of a system is sure to sit at the mean of the locations of the marbles.

## ARCHIMEDES LAW OF THE LEVER

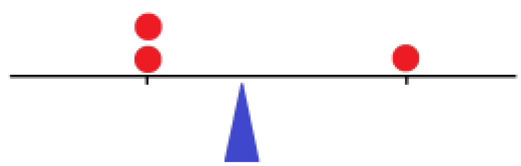
Third-century BCE, Greek scholar Archimedes thought through the subtleties of balance points.

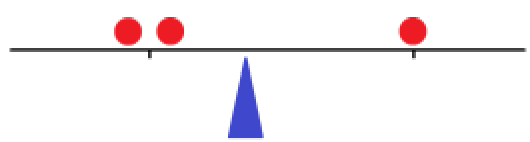
Suppose we have three marbles on a weightless rod, two coinciding at the same location.

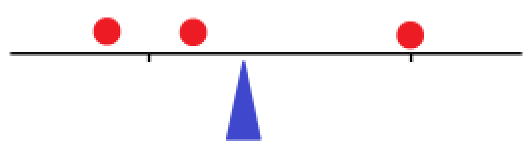


Intuitively we expect the balance point of this system to be closer to the two marbles than the single one. How much closer?

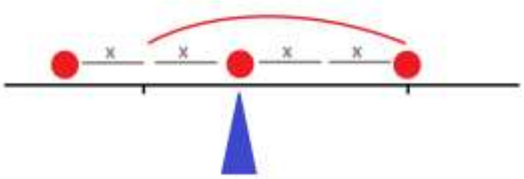
Archimedes realized that two marbles sitting on top of one-another at a single location have the same physical properties as two single marbles equally spaced about that location. So teasing apart the two marbles in a symmetrical fashion about their original location does not change the balance point of the system.







Keep teasing the two marbles apart until we have three equally spaced marbles. The balance point of the system still has not changed.

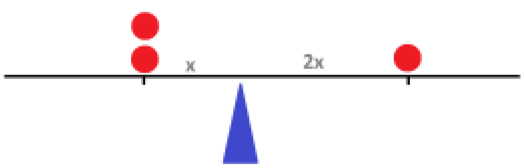



But the balance point of three equally spaced marbles occurs at the location of the middle marble. (A principal that Archimedes assumed was true. It feels right!)

And if each of the two marbles scooted $x$ units to the left and right of their original location, we see that location of the balance point is one-third of the way between the two original locations.

We have:

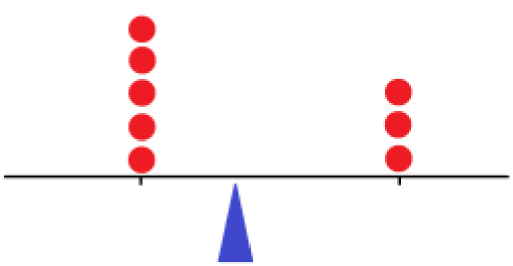
*The balance point of two marbles at one location and one at another occurs at a position that divides the distance between them in a 1:2 ratio.*



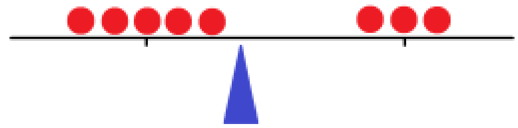

**Comment:** We also have that the balance point of two single marbles occurs at a position that divides the distance between them in a 1:1 ratio.

Let's try a more complicated example.

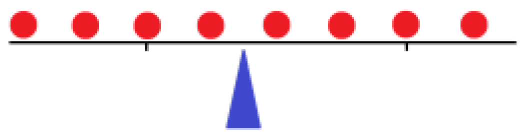
Where is the balance point of a system of eight marbles: five sitting at one location and three at another?



Let's split each stack of marbles symmetrically about each of their respective locations. This does not affect the location of the balance point.



Keep doing this until we have a system of eight equally spaced marbles.



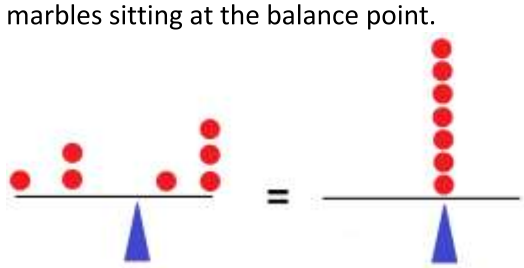
The balance point of an even number of equally spaced marbles is halfway between the center two marbles. We see that this is occurs at the position that divides the distance between the two original locations in a 3:5 ratio.

In general, we have

**Archimedes' Law of the Lever:** *The balance point of a stack of $a$ marbles at one location and a stack of $b$ marbles at another occurs at the position that divides the distance between these two stacks in a $b:a$ ratio.*

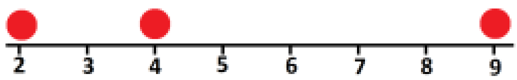
**BACK TO AVERAGES**

Physicists like balance points of systems. They realize that $N$ one-gram marbles sitting on a rod is a system that has the same physical behavior as a stack of $N$ marbles sitting at the balance point.

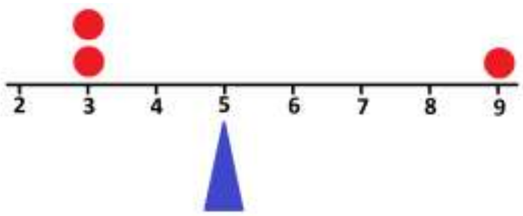



Let's make use of this!

Consider our earlier example of marbles sitting at positions $2$, $4$, and $9$.




The subsystem of two marbles at positions $2$ and $4$ is physically equivalent to a stack of two marbles at their own balance point at $3$.




By Archimedes Law of the Lever, the balance point of this system occurs one-third of the way between locations $3$ and $9$, namely at $5$.

Taking what we just did a little more abstractly: Suppose we have three marbles at positions $x_1$ , $x_2$, and $x_3$.

This is equivalent to a system of two marbles at location $\frac{x_1 + x_2}{2}$ (their own balance point) and one marble at $x_3$.

By Archimedes' Law of the Lever, this system has balance point one third of the between these two locations. That's

$$\frac{x_1 + x_2}{2} + \frac{1}{3}\left(x_3 - \frac{x_1 + x_2}{2}\right).$$

This equals

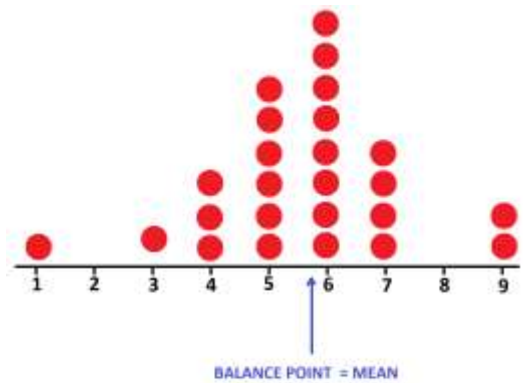
$$\frac{2}{3} \cdot \frac{x_1 + x_2}{2} + \frac{1}{3} \cdot x_3 = \frac{x_1 + x_2 + x_3}{3}.$$

If there is a fourth marble at position $x_4$, then the four-marble system is equivalent to three marbles stacked at location $\frac{1}{3}\left(x_1 + x_2 + x_3\right)$ and one marble at positon $x_4$. Archimedes' Law of the Lever says that this has balance point at

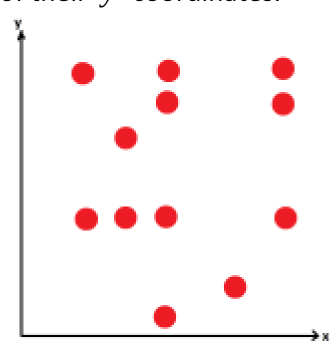
$$\frac{x_1 + x_2 + x_3}{3} + \frac{1}{4}\left(x_4 - \frac{x_1 + x_2 + x_3}{3}\right)$$
$$= \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

In this way, one can indeed establish that the balance point of any system of $N$ marbles on the number line occurs at the arithmetic mean of their locations.

Now I believe pictures like these I see in textbooks!




**CHALLENGE:** Argue that the balance point of marbles scattered in the plane occurs at the location $\left(m_x, m_y\right)$ where $m_x$ is the mean of the $x$-coordinates of all the marble locations and $m_y$ is the the mean of their $y$-coordinates.





tanton.math@gmail.com