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TANTON'S TAKE ON …

A PYTHAGORAS PRECURSOR

MAY 2014

Here's a lovely series of questions to get a class going on Pythagorean thinking. I've presented them here just as I have done in conducting a class discussion.

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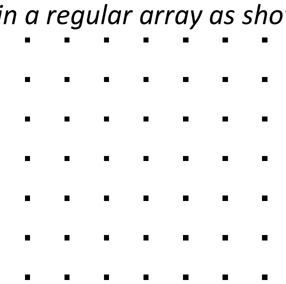
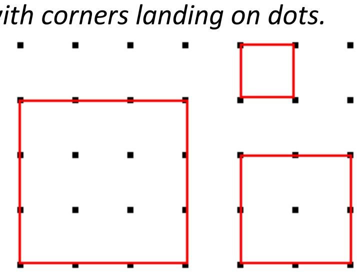
A square lattice is a collection of dots arranged in a regular array as shown:



On such a lattice it is possible to draw squares of areas 1, 4, and 9 square units, each with corners landing on dots.

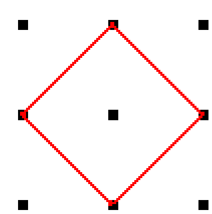


Challenge 1: *Draw a square of area 25 on the lattice.*

Challenge 2: *Draw a square of area 2 on the lattice. (Make sure each corner of the square lands on a dot!)*

This second task is quite a challenge for students. One needs an epiphany! Be sure to give the intellectual and emotional space for epiphanies to occur.

Okay, here's a lattice square of area 2. Do you see that it is composed of four triangles each of area $\frac{1}{2}$?

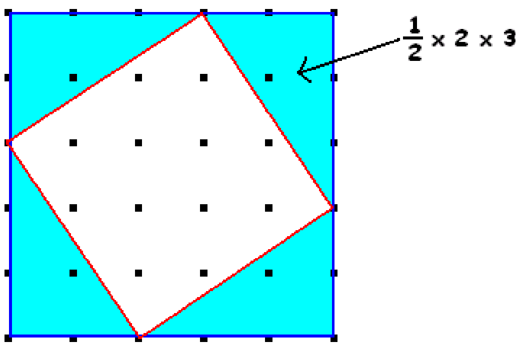


Challenge 3: *Draw a tilted square of area 5 and a tilted square of area 8.*

For each of the numbers 1 through 25 try to draw a lattice square of that area. Which areas are possible?

Students will struggle computing areas of tilted squares. Allow this to become a topic for general discussion. Invariably someone will suggest this alternative approach:

To compute the area of a tilted square, enclose it in a non-tilted square. Compute the area of the large square, and subtract from it the areas of the four triangles that lie in the corners of the large square. Each of those triangles is just half a rectangle.



$$\text{area} = 25 - 4\times\left(\tfrac{1}{2}\times 2\times 3\right)$$
$$= 13$$

This is indeed an easier way to compute the areas of tilted squares. Students will find that the following numbers up to 25, for certain, can be the area of a lattice square:

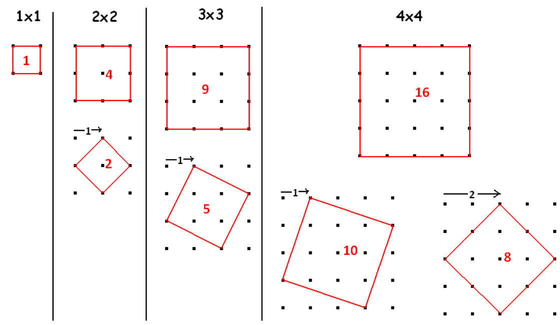
AREA NUMBERS: $1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25$.

This leaves the remaining numbers as questionable.

Questionable Numbers:
$3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24$

Are we certain that our list of area numbers is complete? How can we be sure there is no lattice square of area 3 or of area 24?

This question calls for a systematic approach to our hunt. You might suggest organizing squares via the sizes of the largest outer non-tilted square.

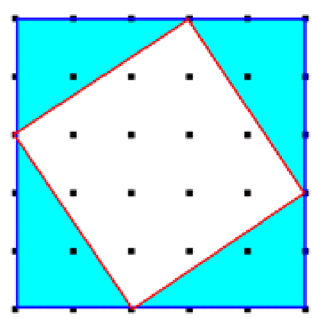


One sees that eventually that the squares in these diagrams are fairly large and that there is no hope of seeing, for example, a square of area 3 or a square of area 6. We learn at least that not all the numbers from 1 to 25 are square area numbers.

Challenge 4: *Show that it is possible to draw a tilted square of area 25. (Presumably you drew a non-tilted version to answer challenge 1.)*

Challenge 5 FOR THE BOLD: *What is the next number after 25 that can be represented in two fundamentally different ways as the area of a square on the lattice?*

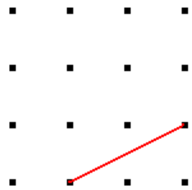
Challenge 6: *Consider again the square of area 13, say.*



$$\text{area} = 25 - 4\times\left(\tfrac{1}{2}\times 2\times 3\right)$$
$$= 13$$

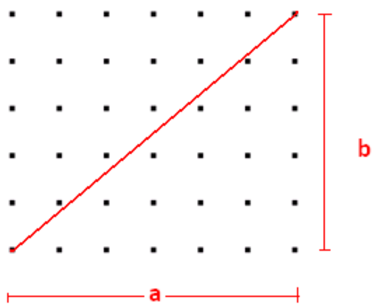
The "root feature" of this square of area 13, that is, its side-length, must be $\sqrt{13}$.

Draw a tilted square and use it to explain why the length of this line segment is $\sqrt{5}$.

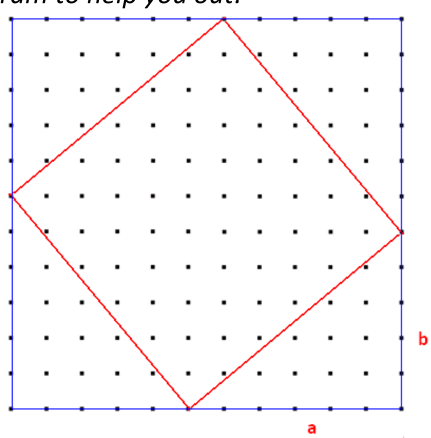


Draw another line segment on a square lattice of length $\sqrt{8}$ *and another of length* $\sqrt{20}$.

Challenge 7: *Find a general formula for the length of a line segment that extends over a "run" of* a *units and a "rise" of* b *units.*



If a hint is needed: *Use the following diagram to help you out:*



The area of the each right triangle is $\frac{1}{2}ab$.

The area of the large square is $(a+b)^2$.

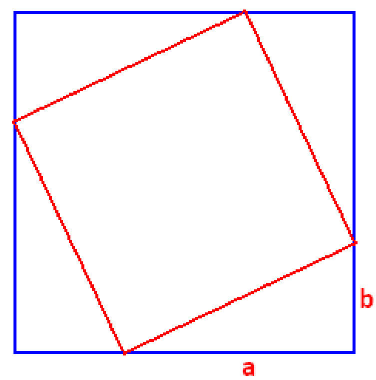
Thus the area of the tilted square is $(a+b)^2 - 4\times\frac{1}{2}ab = a^2 + b^2$ and its side-length is $\sqrt{a^2+b^2}$.

THE AREA NUMBERS

So far we have seen that the numbers $1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25$ *appear as areas of lattice squares. The numbers* 3 *and* 6, *for sure, do not, and we suspect that* $7, 11, 12, 14, 15, 19, 21, 22, 23, 24$ *also fail to be area numbers.*

Can we prove that 24, *for sure, is not an area number?*

Students will likely reason, after challenge 7, that if there is a tilted square of area 24, then $24 = a^2 + b^2$ for some integers a and b.



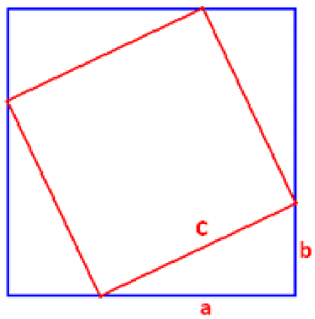
It is easy to check that 24 is <u>not</u> the sum of two square numbers $(1, 4, 9, 16, 25, 36, \ldots)$ and so there is no tilted square of area 24. In fact, one can check that none of the numbers $3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24$ is the sum of two squares, and so our list of area numbers up to 25 is indeed complete.

(Optional) Challenge 8: *List all the area numbers up to* 50.

HARD, OPTIONAL, BUT SURPRISING CHALLENGE: *Prove that the list of area numbers is closed under multiplication. That is, prove that if* m *and* n *are each area numbers, then so is their product* $m\times n$. (For example, 4 and 5 are area numbers, and so is their product 20.)

FINAL PEDAGOGICAL COMMENT

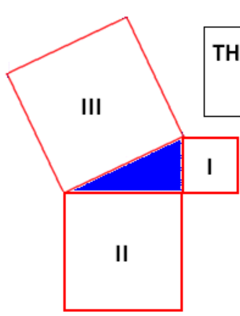
The stage is now set for a discussion of the Pythagorean Theorem and its proof.



The area of the tilted square, c^2, is $(a+b)^2 - 4\times\frac{1}{2}ab = a^2 + b^2$. This picture and the algebraic work for it holds even if a, b, and c are not integers.

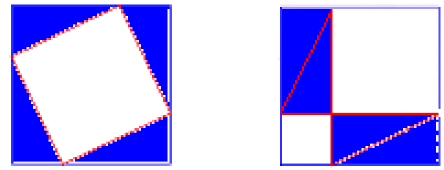
Comment: I personally do not take this step of linking the algebra to a proof of the Pythagorean Theorem. (After all, algebra wasn't invented until a millennium after Pythagoras' time. To ancient scholars, the Pythagorean Theorem was, and still is, a statement of geometry, not algebra!)

Once students are familiar with <u>seeing</u> areas in diagrams, I follow the Chinese Area Proof of the theorem.
(See *GEOMETRY Volume 1*, available at www.lulu.com, for details. Search under "TANTON GEOMETRY.")



THE PYTHAGOREAN THEOREM:
area I + area II = area III

PROOF:



white space = area III   white space = area I + area II

Comment: Challenge 7 can be seen as establishing the distance formula in geometry, if you like. (But do note that the distance formula is just a statement of the Pythagorean Theorem!)

FINAL MATHEMATICAL COMMENTS

An age old question, since the time of Pythagoras (ca 500 BCE) is: *Which numbers are sums of two squares?*

In this essay we have essentially proven: *An integer is the area of a lattice square if, and only if, it is the sum of two squares.* These squares may be identical (a square of area 8 exists because $8 = 2^2 + 2^2$) and one of these squares may be zero (a square of area 25 exists because $25 = 0^2 + 5^2$; it is a non-tilted square).

Pierre de Fermat (1601-1655) is believed to be the first to prove: *An integer is expressible as a sum of two squares if, and only if, each prime in its prime factorization that is one less than a multiple of four appears an even number of times.* The primes that are one less than a multiple of four are:

$$3, 7, 11, 19, 23, 31, 43, \ldots$$

Thus $24 = 2^3 \cdot 3$ is not a sum of two squares, $72 = 2^3 \cdot 3^2$ is, and $2^{324} \cdot 5^{23} \cdot 7^{4002} \cdot 11^2 \cdot 29^{37}$ is as well!

For a proof of this result, if you are feeling game, have a look at Appendix 1 of *MATHEMATICS GALORE!* available at www.maa.org.

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