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## *TANTON'S TAKE ON …*

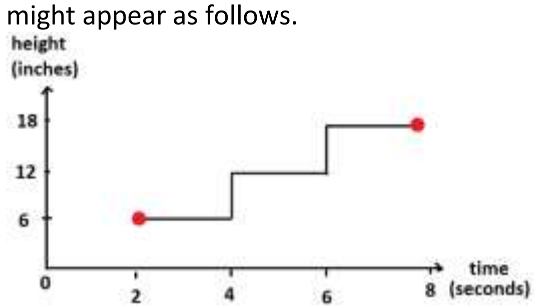
# AVERAGE RATE OF CHANGE

**MARCH 2016**

We adults know too much. And our knowing too much sometimes gets in the way of our curriculum writing, our lesson planning, and our teaching. We often forget that jargon for concepts is sometimes motivated not by the initial concepts themselves, but by the logical consequences of the ideas, from the big connections we see only after from many years of studying and mulling on those concepts. Students seeing new ideas for the first time don't have those years of mulling at hand and so do not have the context for some jargon. But we are often too familiar with the ideas to notice this issue.

Consider the "average rate of change" of a function. This name is confusing! It should be called the "overall rate of change" of the function. That, after all, matches exactly what we do to calculate it.

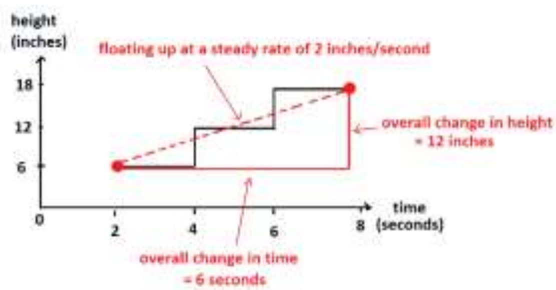
Suppose I walk along a garden path. It has six-inch high steps along its way and I record my height (in inches above the lowest level of the path) as time goes on. A graph of my height with respect to time, from time 2 seconds to time 8 seconds, might appear as follows.




(What is happening at times 4 and 6 seconds? I guess I can leap up steps perfectly vertically in no time at all!)

Asking for the overall rate of change of height over the time period makes good sense: I see that – overall - I rose a total of 12 inches over a period of 6 seconds.

If I started at the left red dot and floated up at a steady rate of $\frac{12}{6}=2$ inches per second for six seconds I would indeed end up at the second red dot.

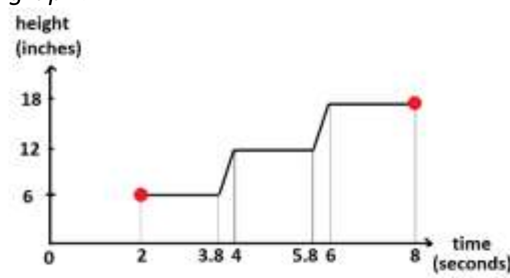



Asking for the "average rate of change" of height over time, however, is confusing.

Computing an average means to take a sum of values and divide by the number of values you have. So "average rate of change" must surely mean we should work out a whole set of rates of change and compute their average? We don't do this when we simply divide the overall change in height by the overall change in time. The name is wrong!

Some texts will use the picture above to argue that the dashed line somehow represents the average slope of the graph. But then again, in the students' minds, why the word "average"? Where is there a sum of values divided by the count of them?

**VAGUE, BUT ILLUMINATING, EXERCISE:** *Here's a slightly more realistic picture of my height versus time graph as I walk up the garden path: there are intervals of 0.2 seconds when I raise my foot from one level to the next. (I still have an overall rate of change* $\frac{12}{6}=2$ *inches per second.) We now see a graph composed of five line segments. Each segment has its own "constant rate of change," called its slope. What is the average value of all the rates of change at all points in this graph?*




**Answer:** This question is indeed vague! There are infinitely many rates of change in this picture, one for each moment of time: at time 2.1 seconds the rate of change is 0 and also at times 2.2, 2.25, 4.77, 6.9887763, and so on, and at time 3.9 the rate of change is $\frac{6}{0.2}=30$, and again a 3.91, 3.99, 5.005884, 7.999989 and so on. We can't sum an infinite number of rates of change to work out their average!

But maybe we can argue this way:

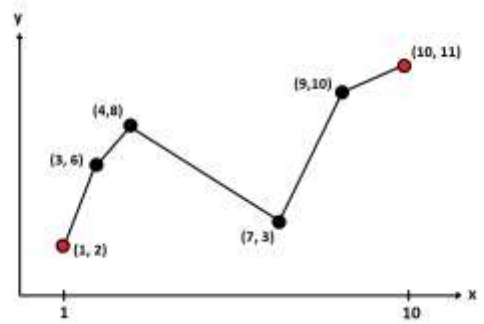
Over the 6 seconds under consideration, from time 2 seconds to time 8 seconds, the rate of change of the graph is 0 for $1.8+1.8+2=5.6$ of those seconds, and 30 for $0.2+0.2=0.4$ of those seconds. So the average value might be computed as:

$$\frac{5.6\times 0+0.4\times 30}{6}=2.$$

(This follows the idea that if I have $56$ bags weighing 10 grams and $4$ bags weighing 20 grams, then the average weight of the bags is $\dfrac{56\times 10+4\times 20}{60}$ grams.)

Is it a coincidence that this average value of the rates of change matches the value of the overall rate of change?

**AN ACTUAL EXERCISE:**
*Consider the function whose graph below is composed of five line segments over the interval from* $x=1$ *to* $x=10$.




*a) What is the overall rate of change of this function over this interval?*

*b) Using the approach in the vague exercise, what is the average value of all the rates of change in the graph?*

[Answer: They both equal $1$.]

## SHOCKER

**THE OVERALL RATE OF CHANGE OF A GRAPH ALWAYS EQUALS THE AVERAGE VALUE OF THE RATES OF CHANGE IN THE GRAPH! (at least for graphs composed of linear sections)**

Working through this second exercise demonstrates why:

The overall rate of change is $\dfrac{11-2}{9}$.

In the graph we have five line segments of slopes $\dfrac{6-2}{2}$, $\dfrac{8-6}{1}$, $\dfrac{3-8}{3}$, $\dfrac{10-3}{2}$, and $\dfrac{11-10}{1}$ over intervals of length $2$, $1$, $3$, $2$, and $1$, respectively, along the $x$ -axis.

Don't do any simplifications!

The average value of the rates of change in the graph is thus.

$$\frac{\text{"sum of all the rates"}}{9}$$

$$=\frac{2\times\left(\frac{6-2}{2}\right)+1\times\left(\frac{8-6}{1}\right)+3\times\left(\frac{3-8}{3}\right)+2\times\left(\frac{10-3}{2}\right)+1\times\left(\frac{11-10}{1}\right)}{9}$$

$$=\frac{\cancel{6}-2+\cancel{8}-\cancel{6}+\cancel{3}-\cancel{8}+\cancel{10}-\cancel{3}+11-\cancel{10}}{9}$$

$$=\frac{11-2}{9}.$$

We see that only the difference of the initial and final heights survives in the numerator, and we get an expression that matches the overall rate of change!

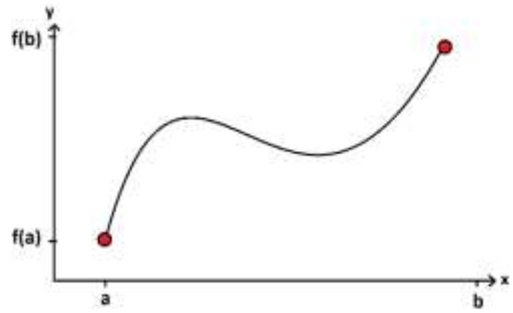
And this is the case for all graphs composed of line segments: we will be working with expressions of the form

$$\frac{(a_1-a)\left(\frac{f(a_1)-f(a)}{a_1-a}\right)+(a_2-a_1)\left(\frac{f(a_2)-f(a_1)}{a_2-a_1}\right)+\cdots+(b-a_n)\left(\frac{f(b)-f(a_n)}{b-a_n}\right)}{b-a}$$

Okay, now I get why overall rate of change is called average rate of change.

**CALCULUS: For those wanting to go deep**

What about graphs not composed of line segments?

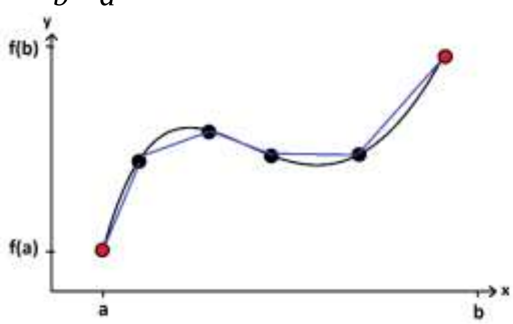



This graph has an overall rate of change. It is $\frac{f(b)-f(a)}{b-a}$ (and this is the slope of the line connecting the beginning and end red dots).

What's its average rate of change?

Well, we can approximate the curve by line segments. And we know from the previous page that that average rate of change for this line segment approximation equals the overall rate of change of the function:

$$\frac{f(b)-f(a)}{b-a}.$$




This is true for <u>any</u> line segment approximation we make. So choose a line segment approximation so "tight" that the slope of each segment basically matches the slope of the tangent line to the curve at each point. The slope of the tangent line to the curve at point $x=a$ is denoted $f'(a)$ in calculus.


Thus we can approximate the sum

$$\frac{(a_1-a)\left(\frac{f(a_1)-f(a)}{a_1-a}\right)+(a_2-a_1)\left(\frac{f(a_2)-f(a_1)}{a_2-a_1}\right)+\cdots+(b-a_{n-1})\left(\frac{f(b)-f(a_{n-1})}{b-a_{n-1}}\right)}{b-a}$$

as

$$\frac{(a_1-a)f'(a)+(a_2-a_1)f'(a_1)+\cdots+(b-a_{n-1})f'(a_{n-1})}{b-a}$$

Taking the limit of a sum of finer and finer approximations in calculus is called an integral. The numerator we have is one of those sums and in taking the limit it becomes the integral of $f'(x)$. The average slope of the curve, in the limit is

$$\frac{\int_a^b f'(x)\,dx}{b-a}.$$

But as every approximation to get to the limit equals $\frac{f(b)-f(a)}{b-a}$ we must have that this limit equals this value too. So it looks like we are concluding:

$$\frac{\int_a^b f'(x)\,dx}{b-a}=\frac{f(b)-f(a)}{b-a}.$$

This is indeed a true statement as we have here the statement of the Second Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b)-f(a).$$

Now I realize it is this Fundamental Theorem of Calculus telling us that the average value of the rates of change of a curve, $\frac{\int_a^b f'(x)\,dx}{b-a}$, must equal the overall rate of change, $\frac{f(b)-f(a)}{b-a}$.

Okay, now I really do get why overall rate of change is called average rate of change!


tanton.math@gmail.com