TANTON'S TAKE ON …

THE PREMISE OF IT ALL

JUNE 2016

Many educators very much like teaching the "box method" for solving quadratics in Algebra II. (See the appendix to this essay for details.) The approach mimics the natural process for solving these equations: it is the method of completing the square – literally! It makes immediate sense and thus requires no formulaic memorization. It is joyful, and it always works. (In fact, use it abstractly to solve $ax^2+bx+c=0$ and out pops the famed quadratic formula.)

The approach teaches a powerful principle in mathematics: *Symmetry is our friend!* Mathematicians will always look for symmetry in a problem and try to use it to their advantage.

Aside: As a simple example, suppose I tell you that I have a rectangle and that the area of this rectangle is 36 square units. What could you tell me about that rectangle? Well, not much. Perhaps it is a 4-by-9 rectangle of area 36, or a 2-by-18 one, or a $7\frac{1}{2}$-by-$4\frac{4}{5}$ rectangle. We have no hope of guessing its dimensions.

But suppose I now reveal to you that that rectangle is symmetrical, that is, that it is a square. When you suddenly know everything there is to know about that figure!

Finding symmetry in a situation provides hope of developing good understanding of the situation.

However, despite the apparent joy of the box method, I often get queries from educators akin to this.

Okay. The box method is great. But many of my students already know the quadratic formula and find it quicker to solve any given problem using the formula instead. And they do have a point! If a quadratic has decimal coefficients, for instance, isn't the quadratic formula indeed just quicker and easier?

Before I give my true answer to this question I do need to point out that the box method will never let you down! It will always work no matter how awkward the numbers might be. (It just might not be fun.)

I also need to point out that one is always empowered to adjust given questions. For example, if given the task of solving

$$2.7x^2 - 0.1x + 9.2 = 7$$

by hand, I would personally choose to first multiply through by ten and work with the equation

$$27x^2 - x + 92 = 70 .$$

Similarly, I'd probably choose to multiply $x^2 + \frac{1}{\sqrt{3}}x - \sqrt{6} = 0$ through by $\sqrt{3}$.

Now my true answer to the question. (And this is indeed a true response: it is the reply I recently gave – tidied up a tad.)

Indeed, if the numbers are awkward and the goal of the day is to just get the numerical answer to the problem, with speed and as little pain as possible, then - absolutely - plugging into the quadratic formula is considerably easier, less time consuming, and less mentally taxing than the box method. In this scenario students should, for sure, absolutely go straight for the quadratic formula. I agree with this point.

But here's the odd thing about all this. The premise of this entire education conversation is wrong!

If the goal truly is just to get to the numerical answer to the problem, then actually the smartest and most accurate thing to do is type the quadratic equation into Wolfram Alpha and press enter! (That is how I would personally solve $2.7x^2 - 0.1x + 9.2 = 7$*.)*

So we're playing this game in the classroom that we're pretending that real technology doesn't exist and that we're back in the 1950s where students must do everything with pencil and paper (with the concession we'll let you use the calculator for the unpleasant arithmetic). It is really quite odd if you think about it, and it is odd that the kids have bought into this too.

In this day and age, where answers are easy to get, I say we must acknowledge that, let it happen, but make the answers incidental and secondary. Our work in the classroom is about the mathematical story of quadratics, the development of ideas, the play of ideas, and the human story of the subject. Take, for example, the problem:

> A river boat travels upstream for 15 kilometers and then returns to start. The river has a steady current of 2 kilometers per hour and the entire return-trip journey took five hours. If the boat's speed was constant throughout the journey (that is, the speed of the boat relative to the water in which is moves is constant), for how long was the boat traveling upstream?

The work here is in developing a meaningful equation to solve! Solving the actual equation is just the dreary grunge work. Wolfram Alpha is best for that.

But changing this perception of why we are teaching mathematics, that actual numerical answers are incidental, is going to take a big shift. The kids have bought into the idea that, in the end, it is the single answer that counts. After all, all our standardized testing is about getting single numerical answers, under speed! That actually worries and saddens me.

Not sure what I advise you say to your kids, except do have a conversation about this strange mental set-up we are following in the classroom, and to be true to your honest human/math teacher self during it. (And perhaps take some steps towards de-emphasizing the focus on numerical answers.)

Comment: If you look at my essay on assessment http://www.jamestanton.com/?p=968 you'll see in my high-school teaching that I often give quizzes and tests with all the numerical answers supplied, but still leaving big blank space below each question for student work. What message do you think this gave my students?

A RADICAL NOTION

I was recently asked about my views on allowing students use of laptops and internet access in the classroom. My answer is surprising. I am all in! And by "all" I mean all! Imagine allowing students access to all mathematics software, all internet materials, all the time - even during exams and quizzes!
What types of mathematics questions would we teachers now ask our students to answer? (This radical idea thrills me!)

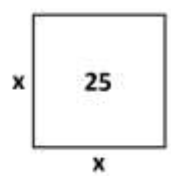
A DASTARDLY QUICK OVERVIEW OF THE BOX METHOD (*See the quadratics course at www.gdaymath.com for full details.)*

The symmetry of a square really is our friend. Solving the quadratic equation

$$x^2 = 25$$

really is asking for the side length of a square of area 25. That side length is $x = 5$. Moving beyond geometry, thinking of this more generally as a problem in arithmetic, we actually see two possible solutions to the equation: $x = 5$ or $x = -5$. Even though negative side lengths make no sense in geometry, the geometric picture still represents arithmetic truth.

$$x^2 = 25$$



So $x = 5$ or $x = -5$.

Solving the quadratic equation

$$(x+3)^2 = 49$$

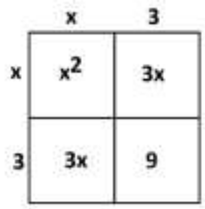
is almost as easy as the previous example: Something squared is 49 so that something must be 7 or -7.

$$x+3=7 \text{ or } x+3=-7.$$

Subtracting 3 throughout gives

$$x=4 \text{ or } x=-10.$$

If we draw a picture of $(x+3)^2$ we see it is a square divided into four pieces.



$$(x+3)^2 = 49$$

$$x^2+6x+9=49$$

If we had been given the quadratic equation

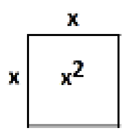
$$x^2+6x+9=49$$

first, we might not have recognized it as a square in disguise. So, is there a way to recognize quadratics as squares in disguise?

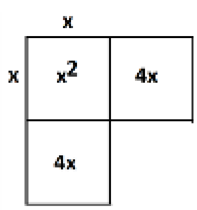
Consider, for example:

$$x^2+8x+16=4.$$

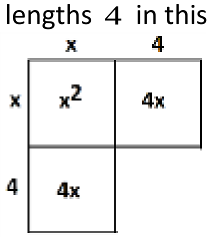
Is it a square? Well, there is certainly an x^2 term that comes from a square.



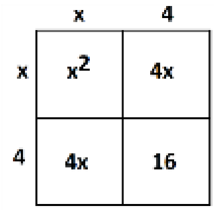
What about the $8x$s? Well, looking at the picture of the top of his column, and working to preserve symmetry (keep it a square!), we think to split the $8x$ into two pieces $4x$ and $4x$ as follows.



Something times x gives $4x$. We must have the side lengths 4 in this picture.



Completing the picture, we see that there is a final piece of area 16



and the picture shows that the equation $x^2+8x+16=4$ is really $(x+4)^2=4$ in disguise. So we have

$$x+4=2 \text{ or } x+4=-2.$$

That is,

$$x=-2 \text{ or } x=-6.$$

Now we have a lovely series of problem-solving challenges to play with!

1. What if I gave you $x^2+8x+15=24$? Draw a picture of a square and you'll see that the number 15 is "wrong"? Oh heavens. What can we do about that?

2. Solve $x^2+3x+5=9$. One can do the box method but one ends up with fractions. Is there a way to carry on with the method and avoid fractions? Can we solve the problem of that middle term being odd?

3. Is $4x^2+12x-3=13$ manageable?

4. What would you do with $3x^2+2x+1=2$?

5. Find a general solution to $ax^2+bx+c=0$ using all the pieces of advice you developed just in case b is an odd number and a is not a perfect square.

[Again … All these details are spelled out slowly and properly in my quadratic notes at www.gdaymath.com.]

tanton.math@gmail.com