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TANTON'S TAKE ON …

A START TO PROBABILITY

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We intuitively believe that for many basic actions – such as flipping a coin, rolling a die, spinning a spinner – that each possible outcome from the action has a certain inherent number associated with it, called its *probability*, and that this number manifests itself as follows:

If a certain outcome of an action has a probability $p\%$ of occurring, then in performing that action many, many times, we'd expect to see that outcome about $p\%$ of the time. (And the observed proportion gets closer and closer to $p\%$ if we perform the action more and more times.)

For example, if we roll a die a million times, we feel we'll likely see a roll of 5 about one-sixth of the time. If we toss a coin a $90,000$ times, we feel we'll see about $45,000$ of the tosses land heads. And so on.

We also feel that this vague intuitive idea works in reverse. For example, if we toss a coin 500 times and it lands heads for a count of 403 of those 500 tosses, then we'll all strongly suspect that the coin is biased (and biased with a probability of about 80% for tossing a head).

Even with just this vague understanding of matters, without a clear definition of probability in hand, we can nut our way through some challenging probability problems just by imagining repeating an experiment a large number times.

Example: *A bag contains 8 Tuscan sunset orange balls and 2 Tahitian sunrise orange balls. The color difference is very subtle and only 70% of people can correctly identify the color of a ball when handed one. Lulu pulls a ball out of the bag at random and tells you over the phone that she pulled out a Tahitian sunrise ball. What are the chances that the ball she holds in her hand really is Tahitian sunrise?*

Answer: Let's assume, as the statistic given suggests, that there is a 70% chance that Lulu can correctly identify the color of a ball when handed one.

Now imagine Lulu conducting this ball-picking experiment a large number of times, say 100 times (and that her chances of correctly identifying colors does not change.)

About 80 of the balls Lulu pulls out will be Tuscan sunset. Of those, she'll identify about $0.7\times 80=56$ of them correctly as Tuscan sunset and 24 she'll incorrectly say are Tahitian sunrise.

About 20 of the balls Lulu pulls out will be Tahitian sunrise, of which $0.7\times 20=14$ she'll correctly identify as such. However, she'll call 6 of them Tuscan sunset.

56 Tuscan sunset Says Tuscan sunset	14 Tahitian sunrise Says Tahitian sunrise
24 Tuscan sunset Says Tahitian sunrise	6 Tahitian sunrise Says Tuscan sunset

Thus in these 100 runs of the experiment we see that Lulu will say "Tahitian sunrise" about $24+14=38$ times and will be correct in saying this 14 of those times. This shows that the probability of her ball really being Tahitian sunrise is

$$\frac{14}{38}\approx 37\%,$$

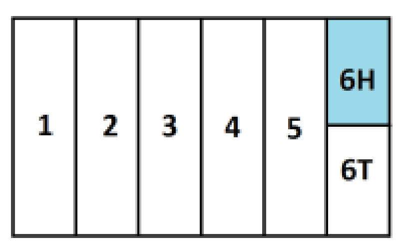
pretty low!

Exercise: *Suppose that one-percent of the population has a certain disease. A test for the disease will return a positive result for 99% of the people with the disease and give a false negative for 1% of those with the disease. The test will correctly give a negative result for 95% of those people without the disease and otherwise gives a false positive.*

You have just tested positive for the disease. Show that there is a one-in-six chance that you actually have the disease. (Is that low?)

PEDAGOGICAL COMMENT:
The fundamental belief offered at the start of this essay is officially called the *Law of Large Numbers.* If explored and used at the very beginning of a probability course, coupled with the area model (drawing a subdivided rectangle with areas in proportion to the percentage of times one expects outcomes), it makes for a powerful and easy student entry into probability thinking and calculating.

Example: In rolling a die and tossing a coin, we'd expect to see a 6 followed by a head about one-twelfth of the time. (This diagram shows the outcomes of running the pair of experiments a large number of times.)



Comment: All is explained and properly explored for K-12 work (and a bit beyond!) in a forthcoming MAA book: *PROBABILITY: A CLEVER STUDY GUIDE.*

THE INFAMOUS BOY-BOY PARADOX

Just to show how powerful the Law of Large Numbers is, let's sort out a famous paradox that befuddled the best of mathematical minds (including Martin Gardner).

Consider three scenarios:

Albert, who you just met, tells you that he has two children and that his oldest child is a boy. What are chances that his other child is also a boy?

Bilbert, who you also just met, tells you that he has two children and that one of his children is a boy. What are the chances that his other child is also a boy?

Cuthbert, also a new acquaintance, tells you he has two children and that one of his children is a boy and was born on a Tuesday. What are the chances that his other child is also a boy?

Assume that giving birth to a child of any particular gender is equally likely and that Bilbert and Cuthbert each secretly flipped a coin to decide which child's gender (and birthday) to reveal.

Here's a table that shows that proportion of men from 196 men with first and second children of various types.

OLDEST CHILD \ YOUNGEST CHILD	Girl	Boy not born on Tuesday	Boy born on Tuesday
Boy born on Tuesday	7	6	1
Boy not born on Tuesday	42	36	6
Girl	49	42	7

(Here's the rational for these numbers:

Among any 196 men, we'd expect, on average 98 men to have oldest child a boy, 98 to have oldest child a girl.

More specifically:
49 men to have oldest child boy, youngest boy; 49 men to have oldest child boy, youngest girl; 49 men to have oldest child girl, youngest boy; 49 men to have oldest child girl, youngest girl.

Also, we expect
$\frac{98}{7} = 14$ men to have oldest child a boy born on a Tuesday; 14 men to have a youngest child a boy born on a Tuesday.

Even more specifically, we expect:

7 men to have oldest child a boy born on a Tuesday, youngest child a girl.
6 men to have oldest child a boy born on a Tuesday, youngest child a boy not born on a Tuesday.
1 man to have oldest child a boy born on a Tuesday, youngest child a boy also born on a Tuesday.

7 men to have youngest child a boy born on a Tuesday, oldest child a girl.
6 men to have youngest child a boy born on a Tuesday, oldest child a boy not born on a Tuesday.
And the same 1 man to have oldest child a boy born on a Tuesday, youngest child a boy also born on a Tuesday.)

In this line of thinking, Albert tells you he is among the 98 men with oldest child a boy. The chances that he is among the subset of 49 men with youngest child also a boy is 50%.

There are $49 + 49 = 98$ men with a child of each gender and we expect half of them, that is, 49 of them, to say "one of my children is a boy" after the flip of a coin. The 49 men with two boys will also say "one of my children is a boy." Bilbert tells us that he is among the 98 men who say these words. The chances that he is among the subset of 49 men with two boys is 50%.

Of the $7+6+7+6=26$ men with just one boy born on a Tuesday, half, that is, 13 will say "one of my children is a boy born on a Tuesday" after the flip of a coin. And the one man with two such boys will say this for certain. Of the 14 men who make this statement, $\frac{1}{2}(6+6)+1=7$ of them have two boys. The chances that Cuthbert has two boys is thus 50%.

So where's the paradox? The chances that each man's other child is a boy is 50% in each case, as you might expect.

Exercise: *Suppose we are now told that Bilbert and Cuthbert did not flip a coin to determine which child's gender (and birthday) to reveal but come from a society in which one must mention that one of your children is a boy if you can, and better yet, must mention that you have a male child born on a Tuesday if you can legitimately do so.*

Show that the probability that Bilbert has two boys is now $\frac{1}{3}$ *and the probability that Cuthbert has two boys is* $\frac{13}{27}$.

Often when this boy-boy puzzle is asked, no information is given as to what mechanism makes a gentleman decide to reveal that one of his children is male, and, as such, the problem is ill-defined and cannot be answered. This is the piece that snags even the most clever of minds (and my mind too when I first worked on this paradox!)

Exercise: *Suppose you know that Bilbert and Cuthbert come from a society that, if they have at least one daughter, would force them to say this. Given what they tell you in the original puzzle, what now are the chances they each have a male second child?*

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