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TANTON'S TAKE ON …

THE EQUATION OF A LINE

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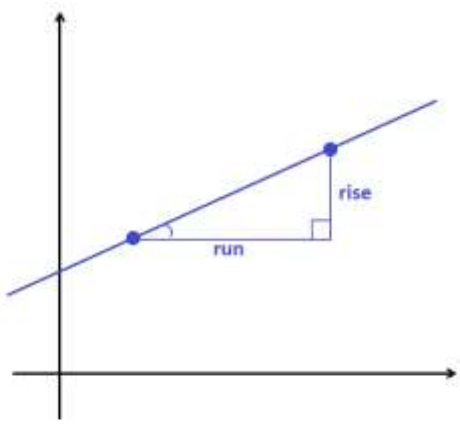
We like to believe that straight lines are the curves in the plane with the property that slope – computed as "rise over run" for two points on the curve – is a constant value among all pairs of points on the curve.



If you believe in the SAS principle of similar triangles, this means that straight lines make a constant angle of elevation to the horizontal. Or, conversely, if you want to take this as the definition of a line being straight and believe in the AA property of similar triangles, then you will conclude that the ratio "rise over run" is a constant value for straight lines.

Question: *Where do vertical lines fit in this discussion of "straight"?*

Suppose I tell you a straight line has slope 7 and passes though the point $(5,8)$. Is there an equation in the variables x and y that must be true for the point (x, y) to lie on the line?

Yes.

For (x, y) to be on the line, the slope computed between the pair of points $(5,8)$ and (x, y) must be 7. This gives

$$\frac{y-8}{x-5}=7,$$

which is an equation that must be true for (x, y) to lie on the line.

But there's a problem: we subtly assumed in our argument that (x, y) is a point on the line different from $(5,8)$. What if $x=5$ and $y=8$? Our derived equation is problematic for these values, despite $(5,8)$ being a valid point on the line.

The standard way to sidestep this problem is to simply rewrite the equation as

$$y-8=7(x-5).$$

This is still an equation that must be true for (x, y) to be a point on the line, and the equation now also holds for the point $(5,8)$ itself.

Question: *Is this a full equation of the line? If x and y are values that make the equation $y-8=7(x-5)$ a true statement about numbers, must (x, y) then actually be a point on the line? What properties of geometry must we assume to say yes to this?*

One notes that any equation of the form $Ax+By=C$, with $B \neq 0$, can be rewritten to look like one of the form $y-p=m(x-q)$, and so is the equation of a line. (For instance, $2x-3y=5$ is algebraically equivalent to the equation $y+\frac{5}{3}=\frac{2}{3}(x-0)$.) If we do allow B to be zero, then this equation reduces to one of the form $x=k$. (For example, $2x+0y=1$ is equivalent to $x=\frac{1}{2}$.) The plot of such an equation is a vertical line. (Ahh! We can incorporate vertical lines in our discussion.)

In short, it seems that the equation of any line, even a vertical one, can be presented in the form

$$Ax+By=C.$$

But note, when asked for the equation of a line, don't be fooled by the use of the word "the." <u>Any</u> equation in variables x and y that must be true for a point (x, y) to be on the line is an acceptable equation. It need not be of the form $y-p=m(x-q)$ or $Ax+By=C$ or $y=mx+b$ or of any other special form with or without a jargony name. It is what you want to do next with the equation that will guide you how you should rewrite it, if at all. (And if there is no next step to a question or a problem, then stop! Leave the equation in whatever form that looks good to you.)

Example: *Find the equation of the line through the points (p,q) and (q, p). (Here p and q are two distinct real numbers.)*

Answer: The slope of this line is -1 and so the equation of the line is $y-q=p-x$.

Of course, many applications in algebra like to give the variables x and y different statuses. Often x is taken as value of some "driving force" quantity in a situation and y is the resultant value of second, dependent quantity. For example, the number of tickets x sold in a raffle determines the revenue y, in dollars, made. In these scenarios it seems helpful to present a linear equation modeling the situation (if a linear model is appropriate) in the form

$$y = \text{some formula in } x.$$

This formula in x will be algebraically equivalent to $mx+b$ for some values m and b.

Example: *A linear function has value* $y=17$ *for* $x=22$ *and value* $y=\sqrt{56}$ *for* $x=\pi$*. Write down an equation for this function.*

Answer: The numbers here are mighty awkward. But can you see that

$$y=17\cdot\frac{x-\pi}{22-\pi}+\sqrt{56}\cdot\frac{x-22}{\pi-22}$$

does the trick? Can you figure out how I constructed this answer with ease? (Put in $x=22$ and then $x=\pi$.) Since nothing more is asked in this question, there is no need to rewrite the answer. What we have here is a valid linear equation, as requested.

ON m AND b IN $y=mx+b$.

According to sources on the internet, such as www.pballew.net/etyindex.html and www.etymonline.com, the word *slope* comes from the IndoEuropean root *sleubh* related to the word "slip." Many words beginning with *sl* in English connect to the idea of slipping or sliding – ground that is on incline is at a *slant*; one slides one's arm through a *sleeve*; lubricants are *slick* and *slimy*. It seems that use of the word *slope* to mean "inclination" was well established by the mid 1600s and, of course, we use the word slope in mathematics to describe the inclination of a line.

There seems to be no clear explanation as to why the letter m is favored in the U.S. to represent slope. French mathematician René Descartes (1596-1650) suggested to the mathematics community that we use letters near the end of the alphabet, x, y, z, w, u, … to denote unknowns, and letters near the beginning of the alphabet, a, b, c, … to represent constants whose values we just don't happen to know. There is no evidence to suggest that he also advised use the letter m for slope. (There is a consistent Urban myth that because the French word *monter*, meaning "to ascend," starts with m we use the letter m.)

Nor is there any clear explanation as to why the letter b is so popular in the U.S. to represent the y-intercept of a line. In fact, these two letter choices are far from universal. School textbooks in other countries depict the slope-intercept form of the equation as

$$y=mx+c$$

(I am told this is common in India. the U.K., and Australia), or as

$$y=kx+m$$

(I am told this is popular in Scandinavia), or as

$$y=kx+n$$

(I am told is popular in Serbia), or vary in their presentation, but always using a pair of consecutive letters of the alphabet:

$$y=ax+b$$
$$y=mx+n$$
$$y=px+q$$

(as popular in some European countries and on calculators that do algebra). Of course, we in the U.S. switch choice of letter for slope when it comes to proportional relationships and write $y=kx$.

So why do we use the letters m and b? Well, we first need to note that the "we" here is the U.S. and the answer is …well, it just seems to have evolved that way. All one needs is one popular textbook author to make a choice and that can set the trend. That is probably what occurred.

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