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TANTON'S TAKE ON …

AVERAGES AS FRACTIONS ADDED WRONG

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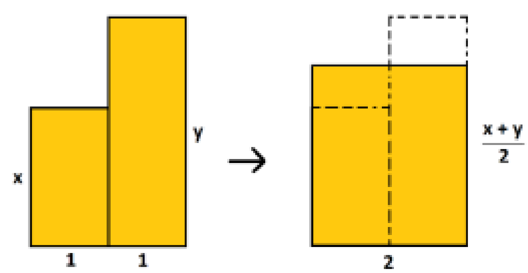
Time for some quirky fun. Consider taking a moment with students to promote playful visual thinking.

The average of two numbers x and y, the number $\frac{x+y}{2}$, is a value that is sure to lie between x and y. We can see this in a (two-dimensional) sandbox:

> Make two piles of sand, each a unit wide, one x units high and the other y units high. Then level out the sand. The result is a pile of height between the two original heights.

Since the total area of sand does not change, the final height is $(x+y)/2$.



CHALLENGE: Using piles and holes show that this result is still true if one, or both, of x or y is negative. (Or zero?)

CHALLENGE: Draw (or just imagine) a picture to show why the average of ten numbers is sure to lie between the highest and lowest of those numbers.

When working with a mix of integers and fractions students often find it helpful to rewrite the integers as "fractions over one."

(For example, 2 is written $\frac{2}{1}$.)

The average of two numbers can be computed using this technique as follows:

> Write x and y each as fractions over one and then add the two fractions incorrectly. That is, blindly add the numerators and blindly add the denominators.
>
> $$\frac{x}{1}+\frac{y}{1} \to \frac{x+y}{1+1}=\frac{x+y}{2}.$$

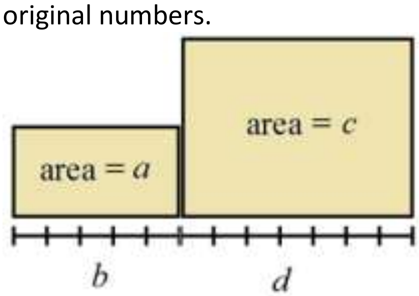
This works too for the average of three or more numbers.

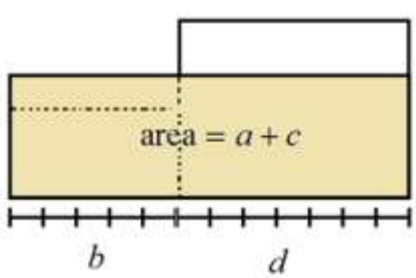
$$\frac{x}{1}+\frac{y}{1}+\frac{z}{1} \to \frac{x+y+z}{3}$$

Incorrectly adding two fractions this way gives what is called the *mediant* of the two fractions.

$$\frac{a}{b}+\frac{c}{d} \to \frac{a+c}{b+d}.$$

The sandbox model shows that the value of the mediant is also sure to lie between the two original numbers.





CHALLENGE: Does this result hold true if one or both fractions are negative or zero?

CHALLENGE: Draw a picture to show that if

$$0<\frac{a_1}{b_1}<\frac{a_2}{b_2}<\cdots<\frac{a_n}{b_n},$$

then

$$\frac{a_1}{b_1}<\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}<\frac{a_n}{b_n}.$$

(Is the condition that the fractions be positive necessary?)

Comment: There is a useful interpretation of the mediant of two fractions: If a team of a girls has b pizzas to share (thus each girl is currently entitled to amount $\frac{a}{b}$ of pizza) meets up with a team of c boys with d pizzas to share (thus each boy is currently entitled to amount $\frac{c}{d}$ of pizza), then, as a big group, each person is entitled to amount $\frac{a+b}{c+d}$ of pizza. (And think about what this means physically: this number can't be worse than the smaller of a/b and c/d, nor can it be better than the biggest of a/b and c/d.)

CHALLENGE: Suppose w_1 and w_2 are two positive numbers that add to 1. Then w_1x+w_2y is a weighted average of x and y. (For $w_1=w_2=\frac{1}{2}$ we get the usual average.) Is there a sandbox picture that proves that the weighted average of x and y is sure to lie between the two numbers?

Is the weighted mediant $\frac{w_1a+w_2c}{w_1b+w_2d}$ of $\frac{a}{b}$ and $\frac{c}{d}$ sure to lie between the two original fractions? (If so, a picture to prove it?)

Do any results, and picture proofs, you obtain here extend to three or more terms?

tanton.math@gmail.com