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TANTON'S TAKE ON ...



MATRIX MULTIPLICATION



FEBRUARY 2014

This content is taken from
<http://www.jamestanton.com/?p=1435>
See the chapter (and videos) there for
full details on this work.

Let me be upfront about my bias: I personally believe that introduction of matrices to the high-school curriculum is forced and unnecessary and, as usually done in the current textbook approach, is damaging to the joyful learning process.

I firmly believe that students really must possess a true sense of ownership of ideas. But the introduction of matrices into the curriculum very much challenges this. Learning and discovery is organic, messy, and non-linear. Matrices are the end result of the learning process, not the beginning. Mathematicians spent decades, if not a century, exploring disparate ideas and struggling with notation for those ideas.

It was a joyous epiphany, a real breakthrough, to find those disparate ideas could be brilliantly united by a clever notational approach, and then to realize that this notational structure has a life of its own! We ruin that joy, and we strip all sense of context for them, by introducing matrices early into the mathematics learning experience.

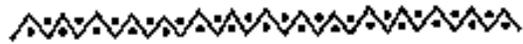
Nonetheless, working with matrices is dictated as part of the high-school curriculum.

In the notes mentioned above I give a detailed account of how I might introduce the full mathematics of matrix work into the high-school classroom. In this essay, I want to just focus on one small part of that discussion: the act matrix multiplication. Multiplying rows by columns in that funny way is strange and mysterious. Why not rows by rows? Or Columns by rows? (Why do we even

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multiply two lists of numbers as a sum of term-by-term products? That, in and of itself, is strange too.)

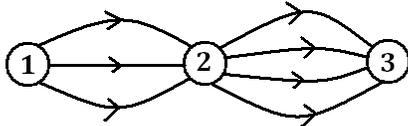
Matrix multiplication, as we are taught to do it, does naturally arise in one particularly accessible context: networks.



NETWORKS

Here's a puzzle from a beginning unit on permutations and combinations:

There are three major highways from city 1 to city 2, and four major highways from city 2 to city 3.



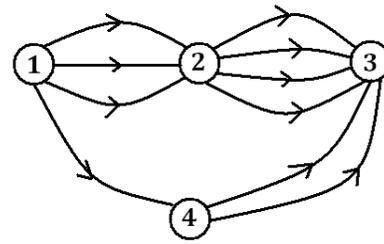
How many different routes can one take to travel from city 1 to city 3?

This puzzle introduces students to the Multiplication Principle for Counting:

If there are a ways to complete a first task and b ways to complete a second, then (assuming no choices made for any one task affect choices made in the other) there are $a \times b$ ways to complete the two tasks together.

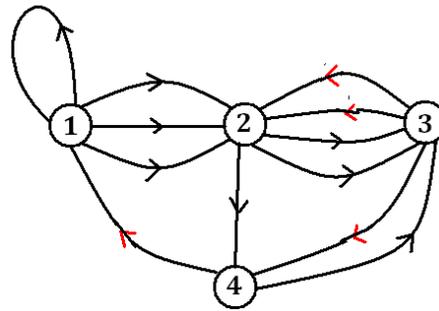
So there are 3×4 routes from city 1 to city 3 in the diagram above.

But if the network diagram were more complicated, the counting becomes more complicated. For example, suppose there is a fourth city:



Now there are now $3 \times 4 + 1 \times 2 = 14$ routes from city 1 to city 3.

But I've made all the connectors one way from left to right. What if the routes can go either direction, and there are even more connections, and maybe even some "loops," and so on?



Here I've changed some of the directions of the routes. Maybe this diagram now represents a flight network for an airline company. (For example, maybe each day the company has two flights from city 2 to city 3, and two flights from city 3 to city 2.) The "loop" from city 1 to itself represents a joy flight you can take!

So counting routes between cities is now quite complicated.

An Exercise Not to Do: How many ways are there to fly from city 1 to city 4 if I wish to fly on a cheap ticket that insists I have two layovers. (City 1 can be a layover after a joy ride!)

For a very large network number of cities and flights the network diagram would be visually complicated,

overwhelming, and hard to analyze. And in counting routes we'd obviously want to program a computer to do all the work for us. But we can't enter a picture into a computer.

But we can encapsulate all the information of a network diagram into a table.

Here's a table for all counts of all the direct routes from city i to city j , for all possible values of i and j .

		To:				
			1	2	3	4
From:	1	1	3	0	0	
	2	0	0	2	1	
	3	0	2	0	1	
	4	1	0	1	0	

I'll call this *table P*, where P stands for "paths," and I'll use the notion P_{ij} for the entry in the i -th row and j -th column of the table, just as people usually do. So:

$P_{12} = 3$, because there are three direct paths from city 1 to city 2.

$P_{34} = 1$, because there is one direct route from city 3 to city 4

$P_{22} = 0$ because there are no routes from city 2 to itself.

and so on.

Given that it is understood that the rows and columns represent cities, we don't really need to write down the borders of the table. We could just write:

1 3 0 0
0 0 2 1
0 2 0 1
1 0 1 0

table P

Mathematicians might even be tempted to just write:

$$P = \begin{matrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$

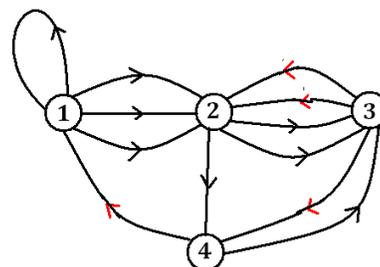
(Notice that I didn't say that they do!)

TERMINOLOGY: Scientists use the term *matrix* for the matter that holds objects of interest in place. For example, a geologist might find a rock sample with garnets in it. She'll speak of the matrix (the background matter) that is holding the garnets.

A table is also an object that holds pieces of interest in place. For this reason, mathematicians call a table of numbers like this a *matrix*. And just so it doesn't look "free floating," they put big parentheses around the values and actually write:

$$P = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Here P_{ij} is the " (i, j) th entry" of P (row i , column j), which, in our example, represents the number of direct routes from city i to city j .



Okay .. so good .. .but so what?

Exercise: How many ways can I travel from city 1 to city 4 with exactly one layover?

Answer: There are four cases to consider:

Layover in city 1:

To do this I need to travel from city 1 to city 1, there are P_{11} ways to do this, and then travel from city 1 to city 4, and there are P_{14} ways to do this. By the multiplication principle there are $P_{11} \times P_{14} = 1 \times 0 = 0$ ways to do this.

Layover in city 2:

To do this I need to travel from city 1 to city 2, there are P_{12} ways to do this, and then travel from city 2 to city 4, and there are P_{24} ways to do this. By the multiplication principle there are $P_{12} \times P_{24} = 3 \times 1 = 3$ ways to do this.

Layover in city 3:

There are $P_{13} \times P_{34} = 0 \times 1 = 0$ ways to do this.

Layover in city 4:

There are $P_{14} \times P_{44} = 0 \times 0 = 0$ ways to do this.

The total number of routes is:

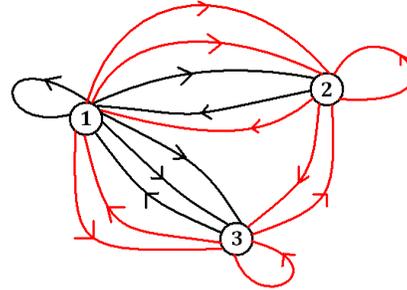
$$P_{11} \times P_{14} + P_{12} \times P_{24} + P_{13} \times P_{34} + P_{14} \times P_{44} = 0 + 3 + 0 + 0 = 3$$

ways to do this.

This example illustrates, in general, that the number of routes from city i to city j with one layover is:

$$P_{i1}P_{1j} + P_{i2}P_{2j} + P_{i3}P_{3j} + P_{i4}P_{4j}.$$

Suppose there are two airlines – “Air is Us” and “Jet Blasters”- which we’ll call airline A and airline B. They each have their own flight networks – A is shown in black, and B in red:



We have path matrices:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise 1:

a) If airline “Air is Us” doubled all the counts of daily flights it currently offers between cities, how would its path matrix change? Mathematicians would call this matrix “ $2A$.” Do you agree that this is appropriate in some sense?

What do you think is the matrix $10A$? What does it mean in terms of the flight network?

b) Ignore the colors in the above network diagram (so pretend the two airlines united and became one company). What would be the combined path matrix be? Why do mathematicians deem it appropriate to denote this matrix as “ $A + B$ ”?

c) Interpret $2A + 3B$. What path matrix is it? For what flight network?

d) “Cheap as Chips” airlines went bankrupt and currently offers no flights anywhere at all. What does its path matrix look like?

PUSHING THE IDEA FURTHER:

Question: How many routes are there between city 1 and city 2 with exactly one layover. Moreover, I insist on flying with “Air is Us” first and then with “Jet Blasters” second.

Answer: Can you see that the answer is $A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$?

Look at the numbers in this answer:

We’ve used $A_{11} = 1$, $A_{12} = 1$ and $A_{13} = 2$, the entries of the first row of A , and the numbers $B_{12} = 2$, $B_{22} = 1$ and $B_{32} = 1$, the entries of the second column of B , multiplied these together in turn, and summed:

$$A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$$

In general:

The number of paths from city i to city j , by first flying with airline A and then with airline B , is the i th row of A “produced and summed” with the j th column of B .

Definition: This construct of “producing and summing” the rows of one matrix A with the columns of a second matrix B occurs often in mathematics. Doing this for each row i of A and each column j of B produces its own table of values:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \dots$$

The new table of values is itself a matrix, called the *matrix product* of A and B . It is denoted as AB .

Exercise 2:

a) For each value of i and j (each either 1, 2, or 3) find the number of one layover routes from city i to city j flying first with “Air is Us” first and then with “Jet Blasters” Verify that you get the answers shown in the matrix product AB shown here:

$$AB = \begin{pmatrix} 3 & 5 & 4 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

b) Find the matrix product BA . What is the meaning of its entries?

Exercise 3:

Let P be the path matrix for the puzzle discussed on page 2. Compute P^2 (that is, compute the product PP). What is the entry $(P^2)_{13}$? What is this number counting?

Also compute P^3 , the matrix product of P^2 and P . What is $(P^3)_{14}$ and what is its interpretation? (Did you do the exercise not to do on page 2?)

Exercise 4: Draw an example of a network with path matrix given by the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 5 \\ 1 & 4 & 1 & 0 & 5 & 0 \end{pmatrix}$$