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TANTON'S TAKE ON …

CONTINUOUS COMPOUND INTEREST

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How does one teach students the compound interest formula Be^{rt} for the final balance after investing B dollars at an interest rate of r % per annum (expressed as a decimal number) for a total of t years? Does one simply present a mysterious formula and say, "Here you go!"?

How does one prove that computing simple interest computed over finer and finer time intervals converges to the peculiar expression $B\left(2.718281828459045...\right)^{rt}$? How did Jacob Bernoulli in 1683 come to conclude there must be an expression of this type? Do pre-calculus students have the tools to follow a derivation?

Comment: Bernoulli recognized there is a constant "e" with a value between 2 and 3 that lies at the heart of understanding compound interest. Decades later Leonhard Euler computed the value of this constant to 18 decimal places and recognized its importance throughout many branches of mathematics. He denoted the constant as e, not after his own name, but simply because e is the next vowel after a, a letter he had already used many times in other papers to represent a numerical value. The letter e has since remained the standard notation for continuous compound interest constant.

ON FIRST INTRODUCING CONTINUOUS COMPOUND INTEREST

One typically introduces this topic with a story akin to the following one.

I wish to invest $1000 for a year. The current going rate, for all banks, is 4% per annum. I look around and find five banks with the following deals:

BANK 1: *Bank with us. We provide simple interest at 4% per annum. Clean and simple.*

BANK 2: *Bank with us. We provide an interest rate of 4% per annum and we compute it monthly.*

BANK 3: *Bank with us. We provide an interest rate of 4% per annum and we compute it weekly.*

BANK 4: *Bank with us. We provide an interest rate of 4% per annum and we compute it daily.*

BANK 5: *Bank with us. We provide an interest rate of 4% per annum and we compute it every hour.*

How much money would I earn with each bank? Which bank offers the best deal for me?

> **Aside:** Banks provide a service: they hold and protect customers' money.
> Don't you find it curious that banks PAY YOU for their service? (What other institutions pay its customers?) Why is that?
>
> Research the history of banking and try to find out when financiers first started paying customers to provide them services.

Let's calculate the total money earned with each bank.

BANK 1: The term "simple interest" means that one is given a stated fraction of one's balance at one specified time. The detail "at a rate of 4% per annum" informs us that we shall earn the fraction $4/100$ of our balance one time at the end of one year.

> **Aside:** Roman emperor Augustus levied a fee of "one part per hundred" on all transactions and exchanges that occurred at auctions. The percent sign % is derived from the Lain term *per cento*, which was translated to *per cento* in Italian, and written in shorthand with a *p* and an *o*. The symbol o/oo is used for *per mille*, one part per thousand.

Thus after one year with bank 1 we'll have in our account the original balance of $1000 plus the fraction $4/100$ of that balance.

$$\begin{aligned} &1000 + 0.04 \times 1000 \\ &= 1000(1+0.04) \\ &= \$1040 \end{aligned}$$

We earn $\$40$.

Comment: We see, in general, that awarding interest on a balance $\$B$, multiplies B by the factor $(1+r)$, where r is the interest rate expressed as a fraction.

New Balance $= B + rB = B(1+r)$.

BANK 2: This bank will divide the interest rate into 12 parts, one for each month, and provide that divided interest on the balance at the end each month.

After the first month, my balance will thus grow to

$$1000 + \frac{0.04}{12} \times 1000 = 1000\left(1 + \frac{0.04}{12}\right)$$

After a second month our balance shall be this amount plus an additional 0.04/12 of that amount.

$$1000\left(1+\frac{0.04}{12}\right)\left(1+\frac{0.04}{12}\right)$$
$$=1000\left(1+\frac{0.04}{12}\right)^2$$

After a third month, the balance shall grow by that same factor again to

$$1000\left(1+\frac{0.04}{12}\right)^3.$$

And so on, for 12 months. My final balance in the account shall be

$$1000\left(1+\frac{0.04}{12}\right)^{12} \approx \$1040.74.$$

I'll earn $\$40.74$ with bank 2. A better deal!

BANK 3: This bank will add the fraction $\frac{0.04}{52}$ to your balance at the end of each week for 52 weeks. (For ease, let's ignore the final partial week of the year.) Thus after one year my balance would be

$$1000\left(1+\frac{0.04}{52}\right)^{52} \approx \$1040.79.$$

I'll earn $\$40.79$. Even better!

BANK 4: Working with a non-leap year we see my balance will be

$$1000\left(1+\frac{0.04}{365}\right)^{365} \approx \$1040.80.$$

(It's actually just under $1040.81.)

Better still!

BANK 5: As there are 8760 hours in a year we see my balance will be

$$1000\left(1+\frac{0.04}{8760}\right)^{8760} \approx \$1040.81$$

(It's actually just over $1040.81.)

> **Question:** What would the balance be if interest was calculated ever minute? Every second? Every nano-second?

It seems as interest is computed and compounded over smaller and smaller time intervals, the final balance increases and, moreover, seems to converge to some ultimate idea value. (Is that ideal value the result of computing interest at each and every instant? What could that mean?)

How can we compute that ideal value?

COMPUTING CONTINUOUS COMPOUND INTEREST

We're looking at the quantity

$$1000\left(1+\frac{0.04}{N}\right)^N$$

for larger and larger values of N. The factor of 1000 is somewhat immaterial in this work and we need not be locked into the number 0.04, so we really want to look at the values of

$$\left(1+\frac{r}{N}\right)^N$$

as N grows.

And to get us going we might as well be kind to ourselves and first try working with a very simple value for r, namely, with $r=1$. Can we say, and <u>prove</u>, anything about the values of

$$\left(1+\frac{1}{N}\right)^N$$

as N grows?

Here are the values of $\left(1+1/N\right)^N$ for $N=2,3,4,5,6$, and 7.

$$\left(1+\frac{1}{2}\right)^2=\frac{3}{2}\cdot\frac{3}{2}=\frac{9}{4}=2.25$$

$$\left(1+\frac{1}{3}\right)^3=\frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}=\frac{64}{27}\approx 2.37$$

$$\left(1+\frac{1}{4}\right)^4=\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{4}\cdot\frac{5}{4}=\frac{625}{256}\approx 2.44$$

$$\left(1+\frac{1}{5}\right)^5\approx 2.48$$

$$\left(1+\frac{1}{6}\right)^6\approx 2.52$$

$$\left(1+\frac{1}{7}\right)^7\approx 2.55$$

It seems the values are increasing and, if you insert larger and larger values for N into $\left(1+1/N\right)^N$ we never seem to get a value larger than 3.

If these two assertions are true, then It is intuitively clear that the sequence of values we are seeing must converge ("crunch up") to some value, call it e, to the left of 3 on the number line, or maybe to the number 3 itself. (University students taking a course in "real analysis" will realize that there is an assumed property of the real numbers at play here.)

But in order to prove this number e exists, we still need to prove our two assertions:

The values of $\left(1+\frac{1}{N}\right)^N$

1. steadily increase as N increases, and
2. never exceed the value 3.

THE GENERAL CONTINUOUS COMPOUND INTEREST FORMULA

Before we prove these two assertions let's complete our analysis of compound interest.

We have, allegedly, that

$$\left(1+\frac{1}{N}\right)^N\to e\approx 2.718...$$

as N grows in value. (Here, the arrow $\to$ is shorthand for "gets closer and closer to.")

Now let's examine $\left(1+\frac{r}{N}\right)^N$. What value does this approach as N grows?

Well, if we loosely phrase our result as

$$\left(1+\frac{1}{big}\right)^{big}\to e$$

then we might think to rewrite $\left(1+\frac{r}{N}\right)^N$ as

$$\left(1+\frac{1}{\left(N/r\right)}\right)^N=\left(1+\frac{1}{\left(N/r\right)}\right)^{(N/r)\times r}$$

$$=\left(\left(1+\frac{1}{\left(N/r\right)}\right)^{(N/r)}\right)^r$$

Now, if N is "big", then so is N/r. So we have here something that looks like

$$\left(\left(1+\frac{1}{big}\right)^{big}\right)^r$$

Our result then says that

$$\left(\left(1+\frac{1}{big}\right)^{big}\right)^{r} \to (e)^{r} = e^{r}$$

for larger and larger inputs. This establishes that

$$\left(1+\frac{r}{N}\right)^{N} \to e^{r}$$

as N grows, and so that

$$B\left(1+\frac{r}{N}\right)^{N} \to Be^{r}.$$

For our opening example, with $B = 1000$ and $r = 0.04$ we see that the "ideal" final balance of my account, computing interest, continuously, over ever possible moment of time (whatever that means) will be

$$1000e^{0.04} \approx \$1040.81$$

fractionally higher than computing interest hourly.

In general: We have just shown that one year of continuous compound interest has the effect of multiplying a balance B by a factor e^{r}. A second year of continuous compound interest will multiply by another factor of e^{r}, and so one's final balance after two years shall be $\left(Be^{r}\right)e^{r} = Be^{2r}$. A third year of continuous compound interest introduces another factor of e^{r} and so will yield the balance Be^{3r}. And so on. After t years of continuous compound interest, the final balance shall be

$$Be^{rt}.$$

This is the formula presented in textbooks.

> **Question:** How can one justify that the formula Be^{rt} is still valid if t represents a fractional count of years?

PROVING THE TWO ASSERTIONS

Now we come to the crunch of matters. How do we prove that that the values of $\left(1+\frac{1}{N}\right)^{N}$ steadily increase as N increases and never exceed the value 3?

This all relies on the combinatorics of expanding brackets and its connection to Pascal's triangle.

$$(x+y)^{0} = 1$$
$$(x+y)^{1} = 1x + 1y$$
$$(x+y)^{2} = 1x^{2} + 2xy + 1y^{2}$$
$$(x+y)^{3} = 1x^{3} + 3x^{3}y + 3xy^{2} + y^{3}$$
$$(x+y)^{4} = 1x^{4} + 4x^{3}y + 6x^{2}y^{2} + 4xy^{3} + 1y^{4}$$

To expand $(x+y)^{4} = (x+y)(x+y)(x+y)(x+y)$, say, we must choose one term from each set of parentheses, multiply them together, and do this for all possible selections of terms, and sum the results.

For example, the term $x^{2}y^{2}$ can appear by selecting x and x and y and y, and again by selecting y and y and x and x; and so on. In fact, the term $x^{2}y^{2}$ will appear as many times as it is possible to arrange two x s and two y s. There are 6 such ways.

$xxyy$

$yyxx$

$xyyx$

$xyxy$

$yxyx$

$yxxy$

One learns in a counting course (see my online notes here) that this count is really $\frac{4!}{2!2!}=6$ the number of ways to arrange 4 letters: 2 xs and 2 ys.

In general, there are $\frac{(a+b)!}{a!b!}$ ways to arrange a xs and b ys, and this leads to the binomial theorem:

$$\begin{aligned}(x+y)^N &= 1x^N + \frac{N!}{(N-1)!1!}x^{N-1}y \\ &+ \frac{N!}{(N-2)!2!}x^{N-2}y^2 \\ &+ \frac{N!}{(N-3)!3!}x^{N-3}y^3 \\ &+ \cdots\end{aligned}$$

We can simplify these coefficients a little

$$\begin{aligned}\frac{N!}{(N-1)!1!} &= N \\ \frac{N!}{(N-2)!2!} &= \frac{1}{2!}N(N-1) \\ \frac{N!}{(N-3)!3!} &= \frac{1}{3!}N(N-1)(N-2) \\ &\vdots\end{aligned}$$

so that when we expand $\left(1+\frac{1}{N}\right)^N$ we obtain

$$\begin{aligned}\left(1+\frac{1}{N}\right)^N &= 1\cdot 1^N + N\cdot 1^{N-1}\cdot\frac{1}{N} \\ &+ \frac{1}{2!}\cdot N(N-1)\cdot 1^{N-2}\cdot\frac{1}{N^2} \\ &+ \frac{1}{3!}\cdot N(N-1)(N-2)\cdot 1^{N-3}\cdot\frac{1}{N^3} \\ &+ \cdots\end{aligned}$$

This equals

$$1+1+\frac{1}{2!}\cdot\frac{N-1}{N}+\frac{1}{3!}\cdot\frac{N-1}{N}\cdot\frac{N-2}{N}+\cdots$$

or equivalently

$$1+1+\frac{1}{2!}\left(1-\frac{1}{N}\right)+\frac{1}{3!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)+\cdots$$

So we have just determined

$$\begin{aligned}\left(1+\frac{1}{N}\right)^N &= 1+1+\frac{1}{2!}\left(1-\frac{1}{N}\right) \\ &+ \frac{1}{3!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right) \\ &+ \frac{1}{4!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right) \\ &+ \cdots\end{aligned}$$

which is a sum with $N+1$ terms.

We can copy this formula and deduce that

$$\begin{aligned}\left(1+\frac{1}{N+1}\right)^{N+1} &= 1+1+\frac{1}{2!}\left(1-\frac{1}{N+1}\right) \\ &+ \frac{1}{3!}\left(1-\frac{1}{N+1}\right)\left(1-\frac{2}{N+1}\right) \\ &+ \frac{1}{4!}\left(1-\frac{1}{N+1}\right)\left(1-\frac{2}{N+1}\right)\left(1-\frac{3}{N+1}\right) \\ &+ \cdots\end{aligned}$$

and since

$$1-\frac{k}{N}<1-\frac{k}{N+1}$$

we have

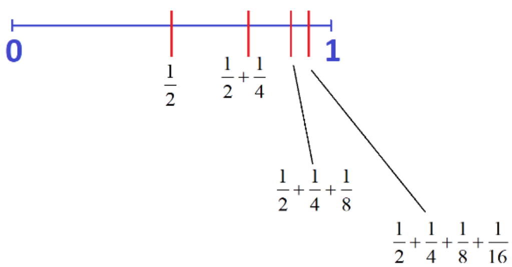
$$\left(1+\frac{1}{N}\right)^N<\left(1+\frac{1}{N+1}\right)^{N+1}.$$

This establishes the first assertion: the sequence of values is indeed strictly increasing. (Sneaky!)

We can also see that

$$\begin{aligned}\left(1+\frac{1}{N}\right)^N &= 1+1+\frac{1}{2!}\left(1-\frac{1}{N}\right) \\ &\quad +\frac{1}{3!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right) \\ &\quad +\frac{1}{4!}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right) \\ &\quad +\cdots \\ &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots \\ &= 1+1+\frac{1}{2}+\frac{1}{3\times 2}+\frac{1}{4\times 3\times 2}+\cdots \\ &< 1+1+\frac{1}{2}+\frac{1}{2\times 2}+\frac{1}{2\times 2\times 2}+\cdots \\ &= 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\end{aligned}$$

The final sum is a sum of a finite number of terms. But it is clear that any finite sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$ is less than 1.



Thus $\left(1+\frac{1}{N}\right)^N$ is sure to be less than $1+1+1=3$.

Our clever way of writing the expansion of $\left(1+\frac{1}{N}\right)^N$ has led to establishing the two claims. We have now proved that the number e exists!

IS THIS APPROPRIATE FOR THE CURRICULUM?

The work here to establish the textbook formula Be^{rt} is hefty, to say the least. Most textbooks choose not to share it. But that leaves the students with nothing to hold on to with regard to playing with continuous compound interest. Some curricula do have students explore the values of $\left(1+\frac{1}{N}\right)^N$ as N grows, but they typically "get murky" in linking this with evaluating $\left(1+\frac{r}{N}\right)^N$ and, in the end, fall back onto the message "just trust us, there is a formula."

I personally think it IS important to show the full mathematics of this work and I have no qualms about giving sessions for students that are purely optional – pay attention or not (with certainly no assessment attached).

Presenting this material in person on a whiteboard is not actually too difficult or overwhelming. I've had success doing this. But it does require working with a group of students who have already had some experience counting words with repeated letters and the binomial theorem and who have fully internalized the lovely and intricate mathematics hidden within Pascal's triangle. (Again, see my counting notes.) So one needs to set matters up months before – and sadly most pre-calculus curriculums don't.

Can we ensure that some meaningful discrete mathematics falls within the early high-school curriculum?

Challenge: Students later encounter in their schooling a number, also called e, in calculus class. It is defined as the base of an exponential function whose derivative equals itself:

$$\frac{d}{dx}e^x = e^x .$$

Is there any reason to believe that this number e is the same number that arises in the study of continuous compound interest? (This is not a trivial question.)

tanton.math@gmail.com