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TANTON'S TAKE ON …

PROPORTION AND RATIO

DECEMBER 2016

Ever since the release of the Common Core State Standards I've been afraid to admit that I don't understand the subtleties of "ratio and proportion," at least, I was under the impression that I don't. This topic has been an issue of much ado in the professional development world for grade-school and middle-school educators the past number of years. (The topic does come up in the high-school curriculum too, but at a level that does not attend to what a ratio or a proportion is in the first place.) I had my own ideas as to the meaning of a ratio or a proportion, but given the worries about these concepts I concluded my thoughts must be too simplistic and missing something deep.

But after some fascinating conversations with some deep thinking colleagues and after finally looking at the K-8 Common Core Standards myself, I think I have nutted out why I have a misconception about having a misconception of ratio and proportion!

I thought the Common Core was using the word *proportion*, a word that I actually don't understand. It doesn't. The Common Core repeatedly uses the phrase *proportional relationship* instead, which emphasizes connection between two quantities, as it should. It rightly removes the hazy use of *proportion* as a stand-alone word.

Just be clear, words that begins with the letters *proportion..* appear in the Common Core Content Standards 15 times and once in a Practice Standard (MP4). The words either come in the phrases *proportional relationship*, *proportional reasoning* or as an adjectival description of some specific thinking or doing action, or in the phrase *constant of proportionality*.

The word *proportion* alone does appear once in the statistics standard HSS-IC-4, but in a different context: the proportion of a population.

Here are some thoughts on what proportional reasoning and ratio mean. This material adapted from content that appears in the Zen Master's Guides to Problem Solving for a middle-school audience. (See https://edfinity.com/products.)

Read the Common Core State Standards for yourself to see if you agree that my thinking outlined here is properly aligned with the intent of the standards.

PROPORTIONAL RELATIONSHIPS

Two quantities appearing in a scenario are said to be in a *proportional relationship*, or just *propotional*, if doubling the amount of one quantity forces the amount of the other to double as well, or tripling the amount of one quantity forces the amount of the other to also triple, or halving the amount of one forces the amount of the other to also halve. And so on.

That is, two quantities are proportional if changing the amount of one of the quantities by some factor forces the amount of the other to change by that same factor too.

For example, suppose a recipe for a flan that serves 5 requires 8 eggs. The two quantities - the number of servings made and the number of eggs needed - are proportional: to double the number of servings we'll need to double the recipe and thus double the number of eggs. To serve 50 people (up by a factor of ten), we'll need 80 eggs, and so on.

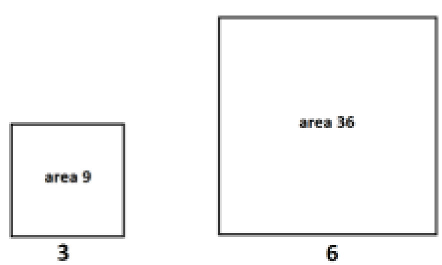
5 servings	↔	8 eggs
10 servings	↔	16 eggs
50 servings	↔	80 eggs

A stack of photocopy paper with 500 sheets is 2 inches high. These two quantities – the height of the stack and the number of sheets in the stack - are proportional: a stack only 1 inch high (half the height) would hold only 250 sheets of paper (half the quantity); a stack 1 ½ inches high (three quarters the height) holds three-quarters the amount of paper, 375 sheets; and so on.

500 sheets	↔	2 inches
250 sheets	↔	1 inch
375 sheets	↔	1½ inches

The number of carrots a person eats one day and the number of bananas she eats on the next day are likely not to be proportional quantities: there is no reason that these two measurements should be linked in any way. And there is no relationship between students' shoe size and performance on a math test, and so on.

Some quantities might be linked to one another, but not in a proportional way. For example, the side length of a square and the area of a square are not proportional quantities: double the first and the other more than doubles.



The number of people needed to build a house and the time it takes for them to do so are not proportional quantities: double the number of people working on the house and the completion time will likely halve, not double.

One has to use everyday knowledge to decide whether or not two quantities described in a scenario are proportional.

Example: *If 6 Martian dollars are worth the equivalent of 7 US dollars, what, in US dollars, is the value of 10 Martian dollars?*

Answer: The quantities of Martian dollars and US dollars are proportional. We have

6 Martian dollars $\leftrightarrow$ 7 US dollars.

It follows that

1 Martian dollar $\leftrightarrow \frac{7}{6}$ US dollars

and so

10 Martian dollars $\leftrightarrow \frac{70}{6} \approx \$11.67$.

Example: *A recipe calls for two-thirds of a cup of maple syrup for every one-and-a-quarter cups of butter. I have one-and-a-half cups of butter and want to adjust the recipe so that I use all of it. How much maple syrup will I need?*

Answer: We have

$\frac{2}{3}$ cups maple syrup $\leftrightarrow \frac{5}{4}$ cups flour

and so

$\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$ C syrup $\leftrightarrow \frac{5}{4}\times\frac{4}{5}=1$ C flour

$\frac{8}{15}\times\frac{3}{2}=\frac{4}{5}$ C syrup $\leftrightarrow 1\times\frac{3}{2}=\frac{3}{2}$ C flour

We need four-fifths of a cup of maple syrup.

INVERSELY PROPORTIONAL RELATIONSHIPS

Two quantities in a scenario are said to be *inversely proportional* if, in doubling the amount of one quantity, the amount of the other halves, or in tripling the amount of one, the amount of the other reduces by a factor of a third. And so on.

In general, two quantities are inversely proportional if changing the amount of one quantity by a factor k causes the amount of the other to change by a factor $\frac{1}{k}$.

For example, if I drive at twice the speed I will complete my journey in half the time: speed and time taken in completing a specific trip are inversely proportional.

In sharing a cake, the amount of cake each person receives is inversely proportional to the number of people sharing the cake: triple the number of people and each person receives a piece reduced to a third of the size.

For an inverse relationship, if we have

a units of quantity one
$\leftrightarrow b$ units of quantity two

then

$2a$ units of quantity one
$\leftrightarrow \frac{b}{2}$ units of quantity two

and

$\frac{1}{4}a$ units of quantity one
$\leftrightarrow 4b$ units of quantity two

and

$\frac{23}{54}\times a$ units of quantity one
$\leftrightarrow \frac{54}{23}\times b$ units of quantity two

and so on.

Again, one has to rely on everyday knowledge to decide whether or not two quantities in a scenario are inversely proportional.

Example: *It takes 3 men 8 hours to wash all the windows of an office building. How many hours would it take 5 men to complete the task?*

Answer: Let's assume that this is an inverse relationship: doubling the number of men halves the amount of time needed, and so on. We have

3 men $\leftrightarrow$ 8 hours.

Thus one man will take triple the time

1 man $\leftrightarrow$ 24 hours

And five men one fifth of this time

5 men $\leftrightarrow$ $\frac{24}{5} = 4.8$ hours.

The answer is 4.8 hours (four hours and 48 minutes).

BOTH TOGETHER

One can combine proportional and inverse proportional relationships within the same scenario.

Example: *If 6 cats can catch 7 rats in 8 hours, to the nearest hour, how long does it take 1 cat to catch 1 rat?*

Answer: We have

6 cats $\leftrightarrow$ 7 rats $\leftrightarrow$ 8 hours

Let's adjust the numbers to 1 cat and then to 1 rat adjusting the time as we go along.

1 cat $\leftrightarrow$ 7 rats $\leftrightarrow$ 48 hours

1 cat $\leftrightarrow$ 1 rat $\leftrightarrow$ $\frac{48}{7}$ hours

This answer is close to 7 hours.

RATIOS

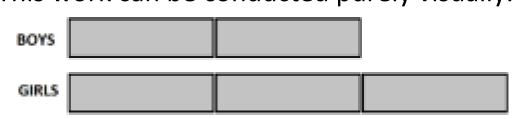
Two quantities in a scenario that are in a proportional relationship are specifically said to be in an *a to b ratio* (often written $a:b$) if whenever a groups of the first quantity occur in a situation, b groups of the second quantity also appear.

For example, suppose in a class of 35 students the ratio of boys to girls is $2:3$. This means we can find a group size x so that two of these groups are boys ($2x$) and three of these groups are girls ($3x$) making 35 students in all. We must have

$$2x + 3x = 35$$

telling us that the group size here is $x = 7$. So there are 14 boys and 21 girls.

This work can be conducted purely visually.



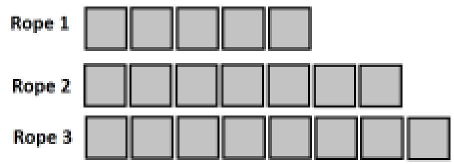
The diagram shows a $2:3$ ratio of boys to girls. There are 35 students in all. Thus each block in the diagram must represent a group of 7 students.

Different block sizes of students give different class sizes possessing the same 2:3 ratio of boys and girls. For example, a class could have $2 \times 5 = 10$ boys and $3 \times 5 = 15$ girls (for a total of 25 students), or $2 \times 40 = 80$ boys and $3 \times 40 = 120$ girls (for a total of 200 students), and so on.

Ratios can extend to more than two quantities.

Example: *I have three sections of rope whose lengths come in a $5:7:8$ ratio. If the total length of rope is 360 meters, what is the length of the shortest piece?*

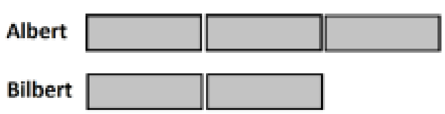
Answer:



This diagram consists of 20 sections of length. As the total length of rope is 360 meters, each section of length is $360 \div 20 = 18$ meters. Thus the shortest rope has length $5 \times 18 = 90$ meters.

Example: *It takes Albert four minutes to pack a goody bag for a party and Bilbert six minutes. Albert and Bilbert started filling goodie bags at the same time and stopped filling bags at the same time. They had, together, filled a total of 60 bags. How many of those bags did Albert fill?*

Answer: Every 12 minutes Albert makes 3 bags and Bilbert 2. So for every three groups of bags Albert packed, Bilbert packed two.



In this diagram, there are five blocks, totaling 60 bags. So each block is a group of twelve. Albert packed 36 bags (and Bilbert 24).

FURTHER DETAILS

Here's a swift overview of some further technical thoughts.

EQUIVALENT RATIOS

If two quantities in a proportional relationship come in an $a:b$ ratio, then we could also say that they also come in a $2a:2b$ ratio or a $13.7a:13.7b$ ratio, or a $\frac{1}{7}a:\frac{1}{7}b$, that is, a $\frac{a}{7}:\frac{b}{7}$ ratio, since scaling the amount of one quantity by some factor means the second has scaled by the same factor.

a units quantity $1 \leftrightarrow b$ unit quantity 2

ka units quantity $1 \leftrightarrow kb$ unit quantity 2

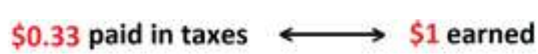
So if Jenny says that a scenario has quantities appearing in an $a:b$ ratio and Sompit says the same quantities appear in a $c:d$ ratio, and both are correct, then it must be that $c = ka$ and $d = kb$ for some number k.

UNIT RATE

If two quantities in a scenario are proportional and come in an $a:b$ ratio and one of the quantities, say the second, is naturally seen as a "driving force" for change in the first, then we might be interested in the *unit rate of change* of that driving force.

For example, if I pay \$33 dollars in taxes for every \$100 dollars I earn, then the number of dollars I earn is seen as dictating the total amount of tax I pay. These two quantities are proportional and come in a $33:100$ ratio.

But we often like to speak of a "tax rate," an amount of money paid per dollar earned. We can rewrite this ratio as the equivalent ratio that focusses on one dollar earned.



We see now a tax rate of 33 cents per dollar earned.

In general, two proportional quantities in an $a:b$ ratio

a of quantity one ⟷ b of quantity two

can be seen as two quantities in an $\frac{a}{b}:1$ ratio.

$\frac{a}{b}$ of quantity one ⟷ 1 of quantity two

We call the number $\frac{a}{b}$ the *unit rate of change* of the scenario (if the second quantity seen as the driving change for the first).

Luckily, if $c:d$ is an equivalent ratio describing the scenario, then we get the same unit rate of change.

$$\frac{c}{d}=\frac{ka}{kb}=\frac{a}{b}$$

Identifying the unit rate of change of a proportional relationship scenario allows one to readily deduce the following.

If there are x units of the "driving force" quantity present in a scenario, then there are $\frac{a}{b}\times x$ units of the other quantity present.

$\frac{a}{b}x$ of quantity one ⟷ x of quantity two

EQUATIONS
If two quantities come in a proportional relationship with ratio $a:b$, then we have just seen that if there are x units of the "driving force" quantity present in a scenario, then there are $\frac{a}{b}x$ units of the other quantity. People often like to denote the amount of the latter quantity present via the symbol y. We thus have $y=\frac{a}{b}x$.

That is, *in any scenario of a proportional relationship between two quantities, the amount x of the driving force quantity present and the amount y of the other quantity present are values that are sure to make the equation $y=mx$ true, where m is the unit rate of change of the relationship.*

How about the converse? If the amounts of two quantities, denoted by x and y, in some scenario are interconnected in some way so as to make an equation of the form $y=mx$ always true for some fixed value m, are those two quantities are in a proportional relationship?

To answer this, we need to ask what happens if we scale the amount of one quantity by a factor k: must the amount of the other scale by that same factor?

Suppose we change quantity x to kx. What amount of the other quantity must we have? We know that "$y=mx$" is true, and this equation must remain true for our new values.

$$??=m(kx)$$

This equation shows that we must have ky units of this latter quantity. We see a quantity also scaled by the factor k.

Suppose instead we change quantity y to ky. Then

$$ky=m(??)$$

shows that we must have kx units of the other, also scaled by a factor k.

We do have a proportional relationship. It is a relationship given by a $y:x$ ratio.

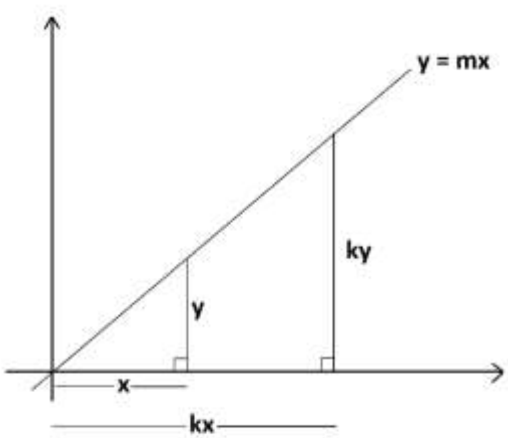
y of quantity one ⟷ x of quantity two

Going a tad further …

Since $y = mx$, we have $\frac{y}{x} = m$ and so we see in addition that m is the unit rate of this proportional relationship.

$\frac{y}{x} = m$ of quantity one ⟷ 1 of quantity two

Comment: If you believe the properties of similar triangles, you can see this uniform scaling at play geometrically.



The basic ideas of proportion and ratio are fairly straightforward and uncomplicated: just follow your nose for a given scenario.

The technical details presented here, usually discussed in the middle-school curriculum, however, bring in a level of abstraction that is different and quite scary. These additional ideas must be introduced slowly and gently.

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