CURRICULUM INSPIRATIONS: www.maa.org/ci

MATH FOR AMERICA_DC: www.mathforamerica.org/dc

INNOVATIVE CURRICULUM ONLINE EXPERIENCES: www.gdaymath.com
TANTON TIDBITS: www.jamestanton.com

TANTON'S TAKE ON …

UNDERGRADUATE MATH COURSES FOR EDUCATORS

AUGUST 2015

In most states, pre-service high school educators must complete a Bachelor's degree in mathematics in order to qualify for teaching certification. The rationale is that advanced mathematics courses help pre-service educators develop the right context and full big picture of the story of mathematics to later be effective in the classroom. Have you indeed found your abstract algebra class, real analysis class, and complex variables class of help in this way? The answer is usually mixed.

I believe that undergraduate educators should be direct with "big picture" help and provide teaching context for their content, and can do so without interrupting the flow of the course. All college math students would benefit from opportunities to look back over mathematics and not just push forward to graduate-level work.

Wouldn't it be lovely if the professor of an abstract algebra class, for example, took the opportunity to give an assignment like the one I show next?

(Just so you can recall, a *ring* is any system of arithmetic with a notion of addition and multiplication associated with it satisfying all the basic rules you would expect. I'll list the rules, and my solutions, after the example.)

EXERCISE:
Consider the ring of integers $\mathbb{Z}$ with the usual operations of $+$ and $\times$.

a) *Using the ring axioms, explain why:*

i) $-(-5)$ *equals* 5.
(The "opposite of the opposite is the original.")

ii) 23×0 *equals* 0.
(The "property of zero.")

iii) $2\times(-3)$ *and* $(-2)\times 3$ *both equal* $-(2\times 3)$.
(One can "pull out" a negative sign.)

In grade school, multiplication is usually seen as repeated addition, at least in the arithmetic of positive whole counting numbers. For example, 4×5 is read and computed as "four groups of five":

$$4\times 5=5+5+5+5=20.$$

This loose definition is not symmetrical (5×4, "five groups of four," is a different computation) and so it is surprising that multiplication is commutative.

b) *How could you explain to a grade-school student why* 137×992 *(*137 *groups of* 992 *) is sure to have the same numerical answer as* 992×137 *(*992 *groups of* 137 *)?*

When negative integers are introduced in grade school, the model of repeated addition is sometimes used to extend the definition of multiplication to them. For example, $2\times(-3)$ is read and computed as "two groups of negative three":

$$2\times(-3)=(-3)+(-3)=-6.$$

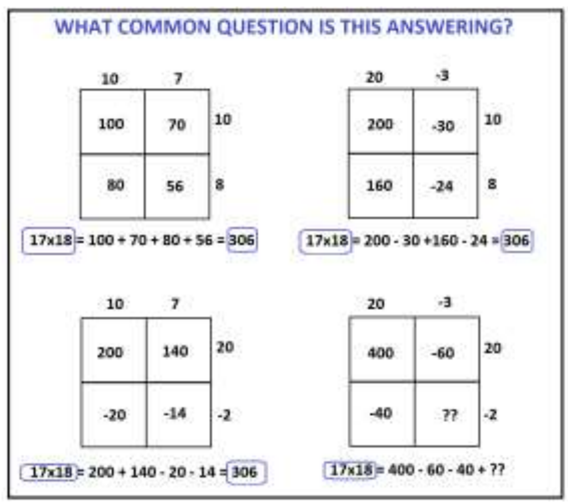
c) *How would you help grade-school students thinking in terms of repeated addition to make sense of the quantity* $(-2)\times 3$*? Must you introduce some ring axioms?*

An age-old school question: **Why is negative times negative positive?**

d) *Using ring axioms, explain why* $(-a)\times(-b)=a\times b$ *for any two integers* a *and* b.

Young students, of course, have not studied abstract algebra and so deserve an answer to this question that is satisfactory to their thinking.

e) *Here is a classroom poster:*



How would you interpret the argument presented in this poster as an alternative ring theoretic proof that $(-a)\times(-b)$ *equals* $a\times b$*?*

THE AXIOMS OF ARITHMETIC

In an abstract algebra course one studies *groups* (systems of arithmetic with just one operation, usually call addition or multiplication), *rings* (systems of arithmetic with two operations, addition and multiplication), and *fields* (systems of arithmetic with three operations: addition, multiplication, and division).

Not all systems with addition and multiplication have multiplication commute. (For example, in the arithmetic of matrices, matrix multiplication generally does not commute.) A system with commutative multiplication is called a *commutative ring*.

The set of integers is a commutative ring: one can add and multiply integers, but one cannot divide integers (and be sure to stay within the system of integers).

Here are the axioms for a commutative ring such as the integers.

Closure: If a and b are elements of the ring, then $a+b$ and $a\times b$ are sure to be elements of the ring too.

Commutativity: $a+b=b+a$ and $a\times b=b\times a$ for all elements of the ring.

Associativity: $a+(b+c)=(a+b)+c$ and $a\times(b\times c)=(a\times b)\times c$ for all elements of the ring.

Zero: There is an element of the ring, denoted 0, with the property that $a+0=a$ for all a. [And $0+a=a$ follows from commutativity.]

One: There is an element of the ring, denoted 1, with the property that $1\times a=a$ for all a. [And $a\times 1=a$ follows from commutativity.]

Additive Inverses: For each element a there is another element of the ring, denoted $-a$, such that $a+(-a)=0$. [And $(-a)+a=0$ follows from commutativity.]

Addition and Multiplication together: $a\times(b+c)=a\times b+a\times c$ for all elements of the ring.

There are some logical consequences of these axioms.

Consequence 1: *There is only one element of the ring that behaves like zero.*

Reason: Suppose 0 and $0'$ both behave as zero. Then $0+0'=0$ since $0'$ is behaving as zero. At the same time $0+0'=0'$ since 0 is behaving as zero. So $0=0+0'=0'$.

Consequence 2: *For each element a there is only one other element that behaves like $-a$.*

Reason: We have $a+(-a)=0$. Suppose $a+b=0$ as well. Then

$$-a=(-a)+0=(-a)+(a+b)$$
$$=(-a+a)+b \quad =0+b=b$$

So b is $-a$.

Consequence 3: $-0=0$.

Reason: $0+0=0$ shows that 0 is behaving as -0. As additive inverses are unique, 0 and -0 must be the same.

ANSWERS TO THE EXERCISE

a) i) Since -5 is the additive inverse of 5 we have $5+(-5)=0$. But this is also saying that 5 is behaving as the additive inverse of -5. There is only one additive inverse, thus $-(-5)$ and 5 must be the same.

ii)

$$23\times 0 = 23\times(0+0)$$
$$= 23\times 0 + 23\times 0$$

Add $-(23\times 0)$ to each side (using the associative property on the right) to see this reads:

$$0 = 23\times 0.$$

In general we can prove $a\times 0 = 0$ for all a .

iii) Is $(-2)\times 3$ behaving as the additive inverse to 2×3?

$2\times 3+(-2)\times 3 = (2+-2)\times 3 = 0\times 3 = 0$.

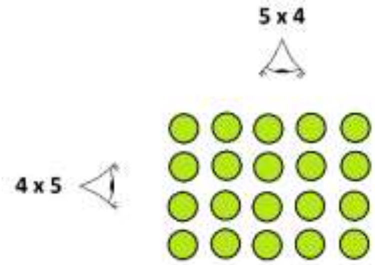
Yes! As there is only one additive inverse, we have $(-2)\times 3 = -(2\times 3)$.

A similar argument gives $2\times(-3) = -(2\times 3)$ as well.

In general we can prove:

$$(-a)\times b = -(a\times b)$$
$$a\times(-b) = -(a\times b)$$

b) Arrange dots in a rectangle.



Look one way and see four groups of five. Look another way and see five groups of four. It is the same picture, so both computations must yield the same result.

This visual works for really big rectangles of dots too.

c) I personally cannot make sense of $(-2)\times 3$. But it feels so compelling to believe that all the rules of arithmetic that hold for positive whole counting numbers should hold for negative numbers too. In particular, $a\times b = b\times a$ just feels natural and right! So if I believe it, then I can say $(-2)\times 3$ is the same as $3\times(-2)$, which is three groups of negative two, -6.

d) Let's show that $(-a)\times(-b)$ is the additive inverse to $-(a\times b)$. (And since additive inverses are unique, $(-a)\times(-b)$ must then be $a\times b$.) By a)iii) $-(a\times b)$ is the same as $(-a)\times b$. So we need to show that $(-a)\times(-b)$ is the additive inverse to $(-a)\times b$. Here goes:

$$(-a)\times(-b)+(-a)\times b = (-a)\times(-b+b)$$
$$= (-a)\times 0 = 0$$

We're good!

ALTERNATIVELY: Using established results

$$(-a)\times(-b) = -(a\times(-b))$$
$$= -(-(a\times b))$$
$$= a\times b$$

e) Writing ab for $a\times b$ and $2a$ for $a+a$ and so on, and skipping over the full details of spelling out use of the associative and distributive laws, we have:

$$a\times b = (2a+(-a))(2b+(-b))$$
$$= 4ab + 2(-a)b + 2a(-b) + (-a)(-b)$$
$$= 4ab + -(2ab) + -(2ab) + (-a)(-b)$$
$$= 0 + (-a)(-b)$$
$$= (-a)\times(-b)$$

	2a	**-a**
2b	4ab	-2ab
-b	-2ab	??

The overall answer must be: axb

tanton.math@gmail.com