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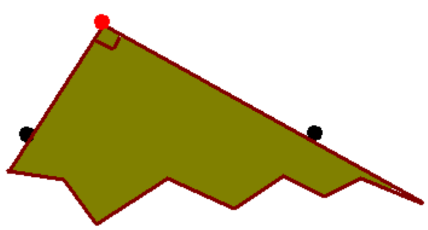
TANTON'S TAKE ON …

CIRCLE THEOREMS

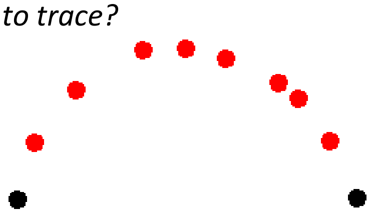
APRIL 2014

Here's a cool activity/puzzle:

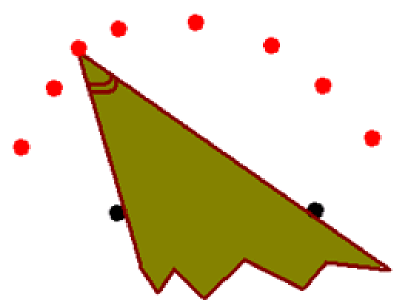
Take a piece of paper and push it up between two nails in the wall. Mark where the 90° corner of the paper lies.



And do this again, pushing the paper up between the nails at a different angle. And then again, another fifty times. What curve do the corners of the paper seem to trace?



Suppose we conducted the same activity but with a piece of paper cut to have a corner angle different from 90°. What shape curve is being traced by this corner?



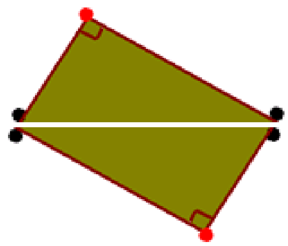
Really do try these activities. (Draw two dots on a white board rather than using nails!)

Can you prove any claims you are tempted to make?

SOME PAPER-PUSHING THOUGHTS

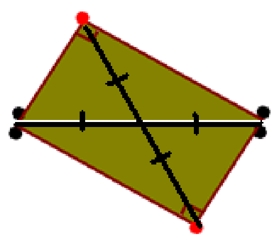
In trying the first paper-pushing puzzle with a 90° corner one is very tempted to say that the curve traced is a semicircle. In fact, one would even say that the center of that semi-circle is the midpoint of the line segment connecting the two nails.

One way to see that this is correct is to draw the right triangle formed by that line segment and the paper, and also a rotated copy of that triangle underneath it.



The two right triangles together make a rectangle. (Why?)

We learn in geometry that the diagonals of any rectangle are congruent and bisect one another.



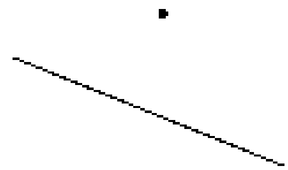
It follows that corner of the paper is sure then to lie on the semicircle with diameter defined by the nails.

Question: Is the answer to the second puzzle also an arc of a circle?

These paper-pushing puzzles can motivate a study of circles and theorems about them.

SOME CIRCLE THEOREMS

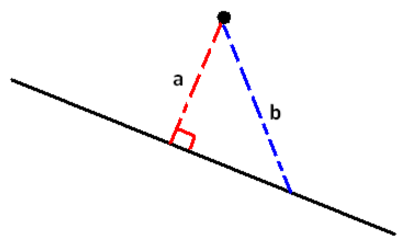
What is the shortest route from a point to line? (Assume the point in question is not on the line.)



Most people would say that the straight path that meets the line at a right angle provides the shortest route.

How can we prove this claim?

One approach is to argue that no other straight route could be shorter. For example, look at the two routes of lengths a and b offered in this diagram:



We see a right triangle and so, by the Pythagorean Theorem we can say:

$$b^2 = a^2 + (something\ else)^2 .$$

Thus b^2 is larger than a^2. And since we are dealing with positive quantities, it follows that b is larger than a. The blue path is indeed longer than the perpendicular red path.

This same argument proves that any straight path to the line different from the red path is sure to be longer in length.

A Philosophical Question: So … has this proved the claim? All we have actually shown is: *Any path that is not the red path is not the shortest path.*

What do you think of this next theorem and its "proof."

Theorem:
1 *is the largest counting number.*

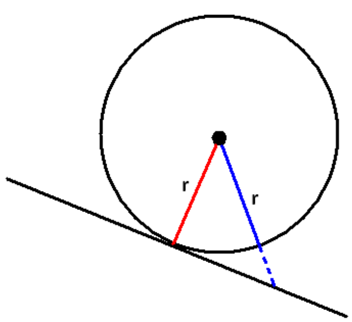
Proof: We'll show that any counting number N that is not 1 cannot be the largest counting number. (This then leaves 1 itself as the only available option as the largest counting number.)

If N is a counting number different from one, then $N > 1$. Multiply through by N to get $N^2 > N$. Thus N^2 is a counting number bigger than N, proving that N isn't the largest counting number.

OUR FIRST CIRCLE THEOREM

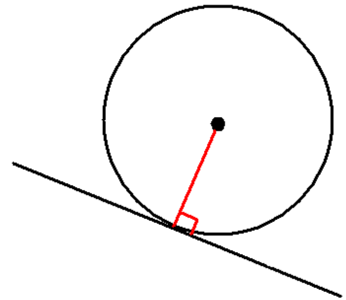
We're all set for our first circle result.

Consider a circle, a tangent line to the circle, and the radius that meets the tangent line at its point of contact.

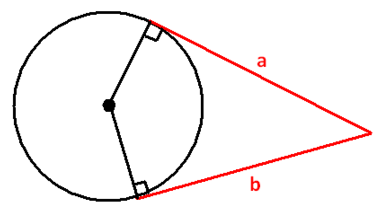


It is clear that any other path from the center of the circle to point on the tangent line is longer than the radius of the circle. Thus the radius segment to the point of contact is the shortest path from the center of the circle to the tangent. By the opening exercise, it must meet the tangent line at a right angle. We have:

Radius/Tangent Theorem:



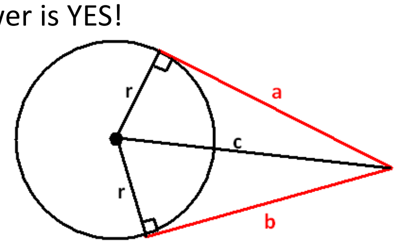
Consider now two tangent lines to a circle, or at least two tangent line segments from a common point outside the circle. The radii to the points of contact meet these tangent segments at right angles.



Exercise: Prove that the four corners of the quadrilateral we see are *con-cyclic*, that is, they all sit on a circle.

It seems compelling to ask: *Does length a equal length b?*

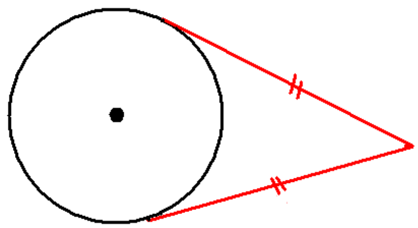
If we draw the line shown, labeling its length c, say, then we see that the answer is YES!



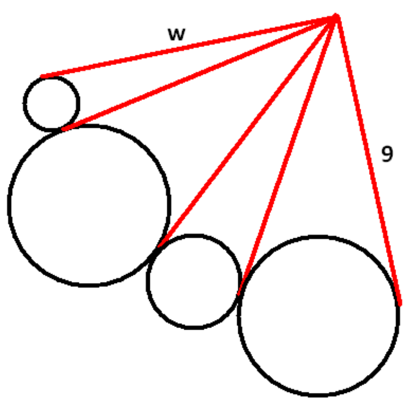
$$a = \sqrt{c^2 - r^2}$$
$$b = \sqrt{c^2 - r^2}$$

We have:

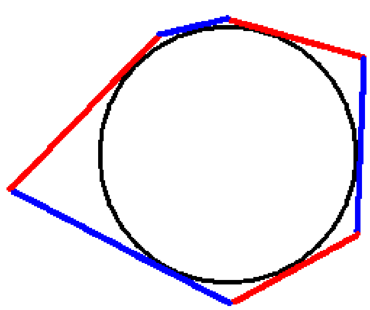
Two Tangents Theorem:



EXERCISE: What the value of w? (Assume we have tangent line segments, tangent at the common points of contact of the circles.)

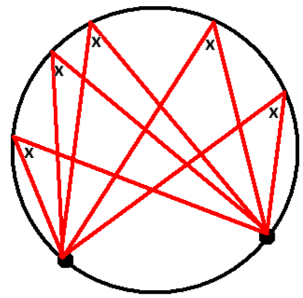


Exercise: In drawing this hexagon that circumscribes a circle (each side of the figure just touches the circle) did I use more red ink than blue ink, or more blue ink than red?



To me, the most astounding circle theorem of all is the following:

THE OPERA HOUSE THEOREM: *All peripheral angles subtended from the same arc have the same measure.*



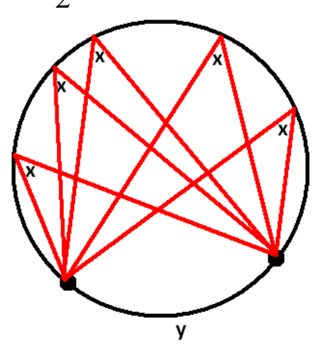
Question: I used old-fashioned language in the statement of the theorem. Can you and your students make good educated guesses as to what "peripheral angles" are and what "subtend" could mean?

Request: Can we teach students the art of deducing the meaning of jargon? (I'd vehemently object to "subtend" and "peripheral" being reduced to vocab words for a quiz.)

Question: Can you guess why my students decided to refer to this result as the "Opera House" theorem?

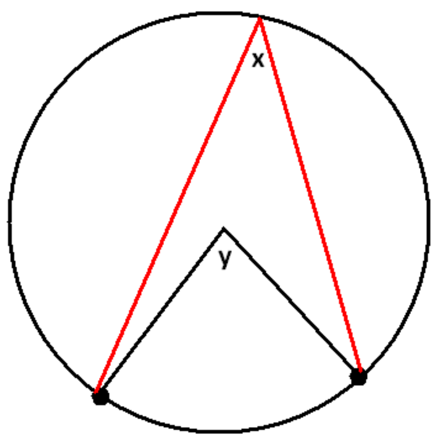
Actually more is true in this famous theorem:

... and common measure of the peripheral angles is half the measure of the arc: $x = \frac{1}{2}y$.

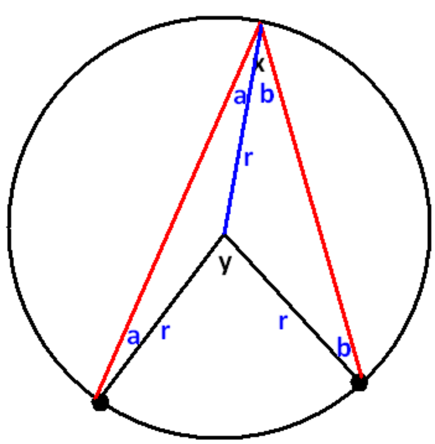


The measure of an arc is simply the "amount of turning" it represents. As an actual angle it can be found as the angle between the two radii of the circle that reach the endpoints of the arc.

Now the challenge: *How might we prove that the measure* x *is half of the measure* y *in the picture below?*



It seems compelling to draw in a third radius. This creates for us isosceles triangles, whose congruent base angles I've labeled a and b.



We see $x = a + b$.

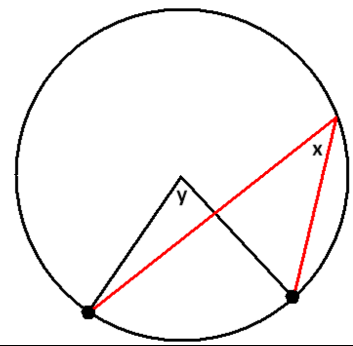
Can we get a formula for y in terms of a and b? You bet!

Look at the three angles around the center point of the circle. These sum to a full 360°. One of these angles is y, another is $180 - 2a$, and the third is $180 - 2b$. We thus have:

$$y + 180 - 2a + 180 - 2b = 360.$$

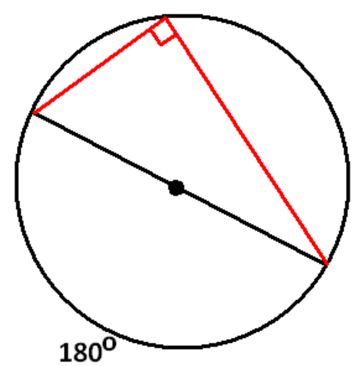
This gives $y = 2a + 2b$ and, indeed, $x = a + b$ is half of this.

Exercise: The picture we drew was too nice. Show that $x = \frac{1}{2}y$ in this lopsided picture too!



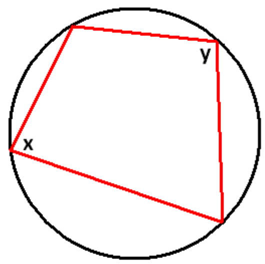
The Opera House theorem has some lovely consequences:

Thales' Theorem: *The angle subtended from a diameter of a circle is a right angle.*



Question: Thales' (ca. 624 – ca. 546 BCE), the "father of geometry," did not use the Opera House theorem to establish this result. What is a much simpler away to prove Thales' theorem? (<u>Hint</u>: Draw in one radius.)

Cyclic Quadrilaterals: *Opposite angles of a quadrilateral inscribed in a circle sum to half a turn.*



$x + y = 180^{o}$

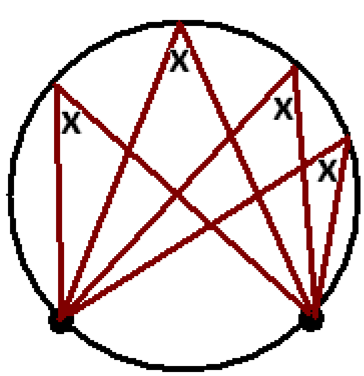
Question: According to the Opera House theorem, why is $2x+2y$ one full turn?

Comment: Since the four angles in any quadrilateral sum to 360°, if one pair of opposite angles are supplementary, then so too is the remaining pair.

THE OPERA HOUSE THEOREM CONVERSE?

If you tried the second paper-pushing experiment it seems that the tip of the paper again traces the arc of a circle.

Now we know from the Opera House theorem that points on the arc of a circle that passes through the two nails subtend the same angle to those two nails.



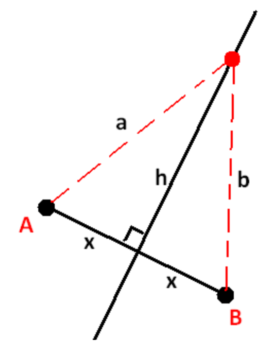
We are now pursuing the converse:

If a curve has the property that points on it subtend the same angle from two fixed points, must that curve be part of a circle?

Here's how to prove this is so.

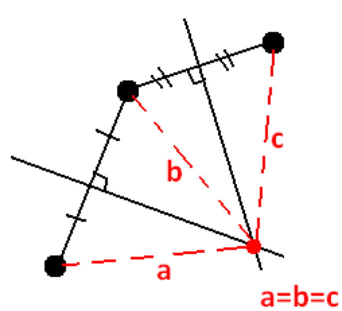
<u>STEP 1</u>: *Any three non-collinear points are con-cyclic.*

First note that any point on the perpendicular bisector of the line segment connecting two points A and B is equidistant from A and B. This is a consequence of the Pythagorean theorem.



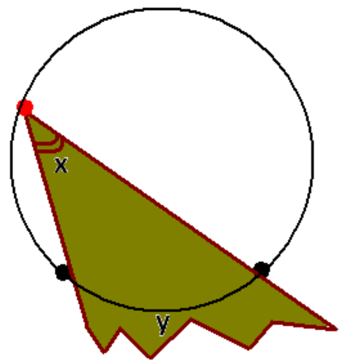
$$a = \sqrt{x^2 + h^2} = b$$

Given three non-collinear points, the point where two perpendicular bisectors intersect is equidistant from all three points.



This intersection point is thus the center of a circle that passes through all three of the given points.

STEP 2: Place the paper up between the two nails one time and draw the circle that passes through the tip of the paper and the two nails.

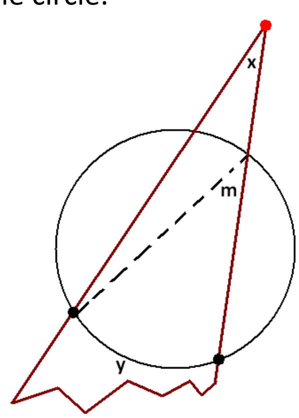


In this picture, x is half of y.

STEP 3: We need to show that if we reinsert the paper up between the two nails a second time, its tip is sure to land on the same circle.

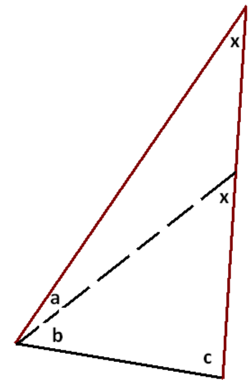
Let's suppose it doesn't and see what goes wrong. There are two cases to consider.

Suppose the tip of the paper lands outside the circle.



If we draw the dashed line shown to highlight angle m, we see, by the Opera House theorem that $m = \frac{1}{2}y$, just like x. This is suspicious!

Draw the chord connecting the two nails. We now have two triangles with the following angle configurations.



We see:

$$b + c + x = 180^\circ$$
$$a + b + c + x = 180^\circ$$

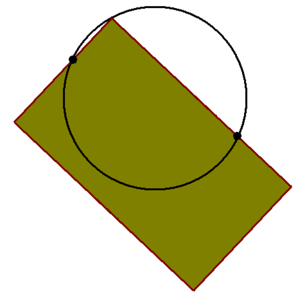
giving $180^\circ + a = 180^\circ$. Oops!

This shows that it can't be the case that the tip of the paper lands outside the circle.

Exercise: Show that a contradiction also arises if we assume the tip of the paper lands at a position inside the circle.

We can only conclude that whenever we reinsert the paper, its tip is sure to land on the same circle. The curve traced by that tip is that circle!

COOL TIP: Suppose you need to find the exact radius of a flower pot. Lay a piece of paper across it as shown. You have now marked off an exact diameter!



Question: How can you use paper, a marker, and string to find the exact center of the pot?

By the way … We are all set to prove the converse of the Cyclic Quadrilateral theorem too:

> **Theorem:** *Suppose a quadrilateral has opposite angles that are supplementary. Then that quadrilateral is cyclic (that is, all four vertices of that quadrilateral sit on a common circle.)*

[Draw the circle that passes through three vertices of the quadrilateral. Why must the fourth vertex sit on that circle as well?]

We'll make good use of this final result in this month's COOL MATH ESSAY.

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