WHY.. It's COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

NOVEMBER 2012

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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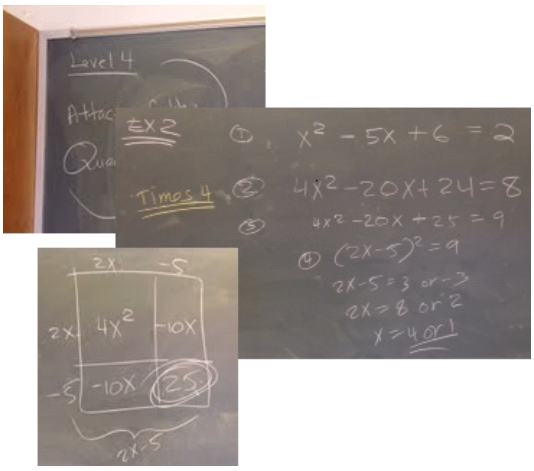
I know there is only one course up at present, but it's a good one and its content is already being used in classrooms across the globe. Check www.gdaymath.com for an online video/text experience on understanding and teaching quadratics the natural and joyful way!

I am hearing from educators in Canada, the U.K., and from all across the United States. I've done a good number of all-day, face-to-face workshops on this content too for folk. Anything to spread joyful math thinking!

The following cool photographs are from James K's classroom in Maryland.



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INTEGER RIGHT TRIANGLES

Each day I am not traveling I post a math puzzle on twitter. (Check out @jamestanton.) And this means each day I am not traveling I wake up in a panic: "What on Earth am I going to post this morning?"

I feel I struck a week's worth of gold this past week as I lay in bed mulling puzzles about right triangles with integer side lengths. Here are some thoughts and ideas from my morning musings:

1. *Prove that the area of right triangle with integer side-lengths is sure to be a multiple of six.*

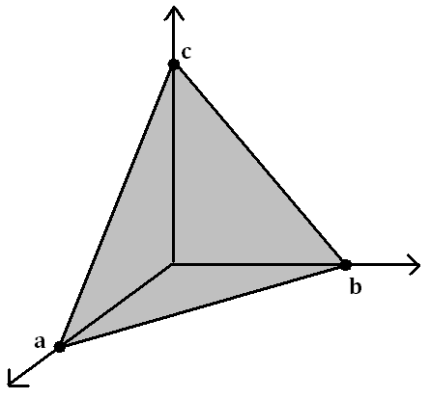
2. *The* $5-12-13$ *right triangle has the property that its area and its perimeter have the same numerical value. Prove that there is only one more integer right triangle with this property.*

3. *Prove that every odd number greater than one* $(3,5,7,9,11,\ldots)$ *can be the side of a right triangle with integer sides. (And every even too?)*

4. *Prove that the radius of the largest circle one can draw inside a right triangle with integer side-lengths is sure to have integer radius!*

5. *If* (a,b,c) *is a Pythagorean triple (so* $a^2+b^2=c^2$*) prove that at least one of* a *or* b *is sure to be divisible by three, at least one of* a *or* b *is sure to be divisible by four, and at least one of* a, b, *or* c *is sure to be divisible by five.*

6. *Are there three positive integers* a, b, c *so that the triangle formed by connecting the point* $(a,0,0)$ *on the* x*-aixs, the point* $(0,b,0)$ *on the* y*-axis, and the point* $(0,0,c)$ *on the* z*-axis has integer sides?*



7. *The triple* $(3,4,5)$ *satisfies*

$$3^2+4^2=5^2.$$

The quadruple $(3,4,12,13)$ *satisfies*

$$3^2+4^2+12^2=13^2.$$

The pentuple $(3,4,12,84,85)$ *satisfies*

$$3^2+4^2+12^2+84^2=85^2.$$

Is there sure to be a hextuple and a spetuple with analogous properties? An n*-tuple of any length I desire?*

Watch out! I am going to give all the answers away in this essay. (Two pages from now in fact.)

First a little general theory …

A CUTE WAY TO FIND PYTHAGOREAN TRIPLES

Did you know that an ordinary multiplication table contains hidden within it an infinite list of Pythagorean triples?

x	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	6	9	12	15	18	21
4	4	8	12	16	20	24	28
5	5	10	15	20	25	30	35
6	6	12	18	24	30	36	42
7	7	14	21	28	35	42	49

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To find one select any two (square) numbers on the main diagonal (say, 4 and 25) and regard them as corners of square in the diagram. Identify the two remaining corners of the square. (10 and 10 in our example.)

Sum the two square numbers, take their difference, and sum the two identical remaining numbers. (Taking their difference doesn't yield much!).

In our example we have:

$$25+4=29$$
$$25-4=21$$
$$10+10=20$$

and we now have a Pythagorean triple:

$$20^2+21^2=29^2.$$

As another example, selecting the squares 9 and 25, with corners 15 and 15 gives:

$$25+9=34$$
$$25-9=16$$
$$15+15=30$$

and

$$30^2+16^2=34^2.$$

Selecting 1 and 4 gives the triple $3^2+4^2=5^2$, and selecting 4 and 9 gives $5^2+12^2=13^2$, and so on.

Question: How do you generate the triple $(7,24,25)$ via this method? The triple $(119,120,169)$?

It is easy to see that this method is sure to always provide a Pythagorean triple:

If we select the two square numbers m^2 and n^2, then the two identical remaining corner entries are mn and mn. We then construct the numbers: n^2+m^2, n^2-m^2 and $mn+mn$, and algebra shows that $(n^2-m^2)^2+(2mn)^2$ does indeed equal $(n^2+m^2)^2$. We have a working triple.

But this method does not produce all Pythagorean triples. (It misses $(9,12,15)$, for example.)

All is not lost however, as this multiplication table approach is sure to produce all the "primitive" Pythagorean triples: those whose three terms have only 1 as a common factor. (So in this sense it doe$(9,12,15)$appear, but in the guise of $(3,4,5)$ with the common factor of three removed.) This was first proved by the great Greek geometer Euclid around 300 B.C.E.

(Forgive me. I am going to be skip his proof here. You can find my explanation of it on page 141 of *Mathematics Galore!* (MAA, 2012), or you can look on the internet. It is a very well celebrated result, easy to find.)

The upshot is that we can construct Pythagorean triples by the formulas.

$$a=n^2-m^2$$
$$b=2mn$$
$$c=n^2+m^2$$

for choices of integers m and n. (And, further: every primitive Pythagorean triple does appear in this way and every non-primitive Pythagorean triple is either of this form too or is a scaled version of a primitive one, scaled by an integer factor.)

We'll make use of this multiplication table method for constructing Pythagorean triples in this essay, but won't make any real use of Euclid's extra step.

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MORE! Choose the four entries at the corners of any rectangle in the multiplication table.

x	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	6	9	12	15	18	21
4	4	8	12	16	20	24	28
5	5	10	15	20	25	30	35
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Call one pair of entries in opposite corners A and B, and the other pair in opposite corners P and Q. Compute their sums and differences. Prove that:

$$(A+B)^2+(P-Q)^2=(A-B)^2+(P+Q)^2.$$

(The example shown produces: $50^2+10^2=34^2+38^2$.) We get integers written as sums of two squares in two different ways!

Comment: In choosing a square on the main diagonal one of the differences is zero. We obtain a Pythagorean triple.

A PRIMITIVE TIDBIT:

Suppose (a,b,c) is a primitive Pythagorean triple, that is, a, b and c share no common factor larger than one and $a^2+b^2=c^2$. Then one of the numbers a or b is sure to be odd and the other even. (And it then follows that c^2 is odd, forcing c to be odd as well.)

Reason: If we rely on Euclid's result, then we can write $a=n^2-m^2$ and $b=2mn$ (or vice versa), which shows at least one of the two is even.

If both a and b are even, then $c^2=a^2+b^2$ is even as well, which means c is even. In this case a, b and c share a common factor of two. Oops!

It must be the case then that precisely one of a or b is odd.

Challenge: It is possible to prove this primitive tidbit without relying on Euclid's result. Can you see how?

In all the work that follows we will assume that the Pythagorean triples under consideration are primitive:

> If an area related to a Pythagorean triple is a multiple of six, then an integer-scaled version of that area is a multiple of six as well. Or if one term of a triple is a multiple of four, then that matching term in a scaled version of the triple is a multiple of four as well. And so on.

All the claims posed in the opening questions hold for scaled versions of the triangles and triples, if we prove they hold for the primitive versions.

GIVING THE ANSWERS AWAY:

No one says you need answer questions in the order they are presented! Let's start with puzzle five.

5. ANSWER: Suppose we have three integers satisfying $a^2+b^2=c^2$. We want to first show that at least one of a or b is divisible by three.

Now every number m leaves a remainder of either 0, 1 or 2 upon division by three and so can be written in the form:

$$m=3k+0$$
$$m=3k+1$$
or $$m=3k+2$$

for some integer k. Squaring gives:

$$m^2=9k^2=3\left(3k^2\right)+0$$

$$m^2=(3k+1)^2=3\left(3k^2+2k\right)+1$$

or

$$m^2=(3k+1)^2=3\left(3k^2+4k+1\right)+1.$$

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This shows that no square number is two more than a multiple of three.

Now … In $a^2 + b^2 = c^2$ can it be that neither a nor b is a multiple of three? If this were the case, then a^2 is one more than a multiple of three, as is b^2, showing that c^2 is a square number two more than a multiple of three. This is not possible. At least one of a and b must be divisible by three then.

Comment: This argument can be more swiftly presented in the language of "modular arithmetic." See chapter 12 of *THINKING MATHEMATICS! Vol 2* available at www.lulu.com.

Let's next tackle divisibility by five. The following work shows the possible remainders of square numbers upon division by five.

$$(5k+0)^2 = 5(5k^2) + 0$$

$$(5k+1)^2 = 5(something) + 1$$

$$(5k+2)^2 = 5(something) + 4$$

$$(5k+3)^2 = 5(something) + 4$$

$$(5k+4)^2 = 5(something) + 1$$

In the equation $a^2 + b^2 = c^2$ it can't be that a^2, and b^2, and c^2 leave only remainders of 1 and 4 upon division by five: at least one has to represent a remainder of zero. (A remainder five is the same as a remainder of zero.) Thus at least one of a, b, or c is a multiple of five.

Divisibility by four is a bit sneaky. It requires looking at remainders upon division by eight.

The following shows that any number squared leaves a remainder of 0, 1 or 4 upon division by eight:

$$(8k+0)^2 = 8(8k^2) + 0$$

$$(8k+1)^2 = 8(something) + 1$$

$$(8k+2)^2 = 8(something) + 4$$

$$(8k+3)^2 = 8(something) + 1$$

$$(8k+4)^2 = 8(something) + 0$$

$$(8k+5)^2 = 8(something) + 1$$

$$(8k+6)^2 = 8(something) + 4$$

$$(8k+7)^2 = 8(something) + 1$$

In particular, odd squares leaves a remainder of 1.

Now look at a primitive Pythagorean triple with $a^2 + b^2 = c^2$. If a is odd and b is even, say, then a^2 leaves a remainder of 1 upon division by eight, and b^2 a remainder of 0 or 4. As c^2 can't have a remainder of 5, it must be that b^2 leaves a remainder of 0. This means that b is of the form $8k+0$ or $8k+4$, and hence is a multiple of four for sure!

Comment: The multiples of 3, 4 and 5 can be "spread apart" or can coincide. Consider the triples $(33, 56, 65)$, $(5, 12, 13)$ and $(119, 120, 169)$, for instance.

1. ANSWER: If a, b and c are the sides of an integer right triangle with hypotenuse c, then $a^2 + b^2 = c^2$, and at least one of a or b is divisible by 3 and at least one is divisible by 4. This ensures that their product ab is divisible by 12. Thus the area of the right triangle, $\frac{1}{2}ab$, is certain to be a multiple of six.

RESEARCH CORNER 1: Does every multiple of six occur as the area of some integer right triangle?

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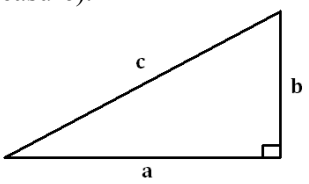
2. ANSWER: Let's find all right integer triangles with area and perimeter possessing the same numerical values (for some chosen unit of measure).



We need to understand which integers from a Pythagorean triple satisfy:

$$\frac{1}{2}ab = a+b+c\,.$$

Rewriting and squaring gives:

$$\left(\frac{1}{2}ab - a - b\right)^2 = c^2 = a^2 + b^2\,.$$

Expanding and manipulating eventually yields:

$$b = \frac{4a-8}{a-4}\,.$$

We need this to be an integer. Let's rewrite this as:

$$b = \frac{4a-16+8}{a-4} = 4 + \frac{8}{a-4}\,.$$

This shows we need $a-4$ to be a factor of eight.

$a-4=1$ gives $a=5$, $b=12$ and we have the $5-12-13$ triangle.
$a-4=2$ gives $a=6$, $b=8$ and we have the $6-8-10$ triangle.
$a-4=4$ gives $a=8$, $b=8$ and we have the $6-8-10$ triangle again.
$a-4=8$ gives $a=12$, $b=5$ and we have the $5-12-13$ triangle again.

Thus the $5-12-13$ and the $6-8-10$ triangles are the only integer right triangles with this property.

RESEARCH CORNER 2: Are there non-right integer triangles with this property? Can you classify them in some way?

3. ANSWER: Every odd number greater than one is the difference of two squares. For example,

$$3 = 4-1$$
$$5 = 9-4$$
$$7 = 16-9$$

and in general

$$2m+1 = (m+1)^2 - m^2\,.$$

So by our multiplication table method we can indeed use it as the leg of a right triangle:

$$a = (m+1)^2 - m^2$$
$$b = 2m(m+1)$$
$$c = (m+1)^2 + m^2$$

Further: Notice that

$$c = 2m^2 + 2m + 1 = b + 1$$

so we have here infinitely many examples of integer right triangles with hypotenuse one longer than a leg. Choosing $m = 1, 2, 3, \ldots$ in turn gives the triples: $(3,4,5)$, $(5,12,13)$, $(7,24,25)$, $(9,40,41)$, $(11,60,61)$, …
(And if we use Euclid's result we can prove that every Pythagorean triple with this property appears in this list.)

Notice that the numbers for the longest legs in these examples, 4, 12, 24, 40, 60,..., are the triangular numbers quadrupled.

Question: So what about the even numbers? Can every even appear as the side of an integer right triangle?

Tough Challenge: The triples $(3,4,5)$, $(20,21,29)$ and $(119,120,169)$ have legs that differ by one. Are there infinitely many Pythagorean triples like this?

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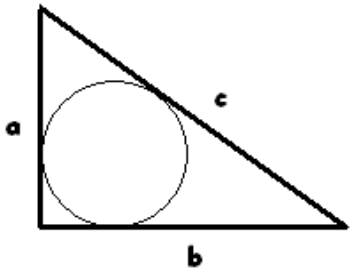
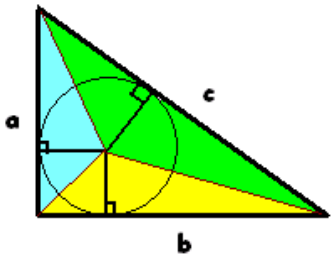
4. ANSWER: Consider the largest circle drawn in an integer right triangle.



Three radii divide the triangle into three subtriangles.



By the "radius/tangent theorem" from geometry each radius is perpendicular to the side it touches and so represents the altitude of a subtriangle. If each radius has length r then the area $\frac{1}{2}ab$ of the triangle can be computed as:

$$\frac{1}{2}ab = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr .$$

This gives:

$$r = \frac{ab}{a+b+c} .$$

Multiplying numerator and denominator each by $a+b-c$ produces:

$$r = \frac{ab(a+b-c)}{(a+b)^2 - c^2}$$

$$= \frac{ab(a+b-c)}{2ab} = \frac{a+b-c}{2}$$

For a primitive triple, one of a or b is odd, the other is even, and c is odd. This means that $a+b-c$ is even, and so r is sure to be an integer. (For a non-primitive triangle, r is a scaled version of an integer and so again is an integer.)

7. ANSWER: We saw in our answer to part 3 that any odd number greater than one can be used as the start of a Pythagorean triple:

a = the given odd
b = even
c = necessarily odd as well

and

$$a^2 + b^2 = c^2 .$$

Now c is an odd number, so we can use it as a leg of a right triangle as well and construct:

c
d = even
e = necessarily odd

with

$$c^2 + d^2 = e^2 .$$

We can also use e as a leg of a right triangle and construct:

e
f = even
g = necessarily odd

with

$$e^2 + f^2 = g^2 .$$

And so on.

So what do we have? A series of Pythagorean triples that look like:

$$0^2 + a^2 = a^2$$
$$a^2 + b^2 = c^2$$
$$c^2 + d^2 = e^2$$
$$e^2 + f^2 = g^2$$
$$g^2 + h^2 = i^2$$

And we can extend this list as far as like.

Now look:

$$a^2 + b^2 = c^2$$
$$a^2 + b^2 + d^2 = c^2 + d^2 = e^2$$
$$a^2 + b^2 + d^2 + f^2 = c^2 + d^2 + f^2 = e^2 + f^2 = g^2$$
$$a^2 + b^2 + d^2 + f^2 + h^2 = \cdots = i^2$$

and so on. We can construct Pythagorean n-tuples of any length we desire.

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Starting this method with $a = 3$ gives:

$$3^2 + 4^2 = 5^2$$
$$3^2 + 4^2 + 12^2 = 13^2$$
$$3^2 + 4^2 + 12^2 + 84^2 = 85^2$$
$$3^2 + 4^2 + 12^2 + 3612^2 = 3613^2$$

6. ANSWER: Yes. But finding numbers like these is mighty hard! $a = 44$, $b = 117$ and $c = 240$, for example, work.

Challenge: Prove that the area of any integer triangle constructed this way is also sure to be a multiple of six!

OPEN RESEARCH:
From question 5 we know that if a, b and c are the sides of an integer right triangle, then their product abc is sure to be a multiple of 60.

1. Not every multiple of 60 appears as one of these products. Which do?

2. Can two different integer right triangles give the same product abc?

This second question is currently an unsolved research problem!

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