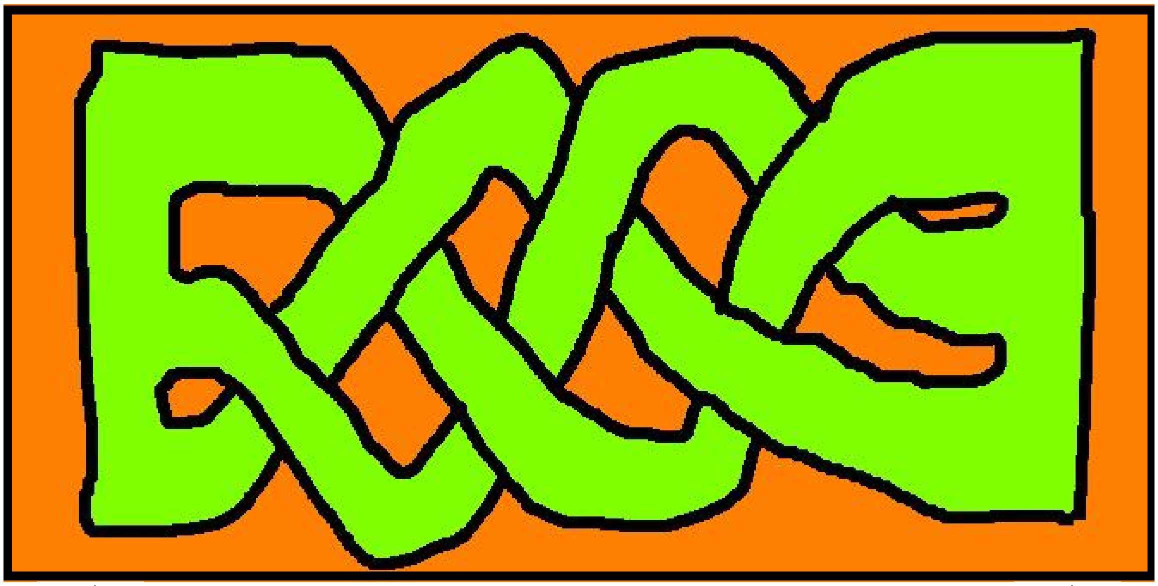


Weally COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

The Sixth Math Letter

SEPTEMBER 2012

Here are three counting puzzles. I thought it might be fun to start with their wordless versions. Can you figure out what the question is in each of these puzzles? (Word-full versions appear on the next page.)

MATH WITHOUT WORDS 1:

1	•
2	• • •|•
4	• • • •|• • • •|• •|•|•
??	

MATH WITHOUT WORDS 2:

?	•
1	(• •)
2	(•(• •)) ((• •)•)
5	(•((• •) •)) (•(• (• •))) ((• •)(• •)) (((• •) •) •) ((• (• •))•)
??	

MATH WITHOUT WORDS 3:

2	• (•)
5	• • (•)• •(•) (•)(•) (• •)
13	• • • (•)• • •(•)• • •(•) (• •)• •(• •) (• • •) (•)(•)• (•)•(•) •(•)(•) (•)(• •) (• •)(•) (•)(•)(•)
??	

PUZZLE ONE:

1	•
2	•• •|•
4	••• •|•• ••|• •|•|•

etc.

In how many ways can one insert dashes in some or all of the spaces between N dots lined up in a row (at most one dash per space)?

PUZZLE TWO:
Curve the dashes and make them parentheses.

?	•
1	(••)
2	(•(••)) ((••)•)
5	(•((••)•)) (•(•(••))) ((••)(••)) (((••)•)•) ((•(••))•)

etc.

In each of these pictures each pair of parentheses contains precisely two objects: either a pair of dots, or a dot and another set of parentheses, or two sets of parentheses. Also, every dot lies within some set of parentheses or nested parentheses.

a) Verify that there are 14 ways to place parentheses around a line of five dots.

b) Begin listing the ways to place parentheses around six dots, but do this in a manner that makes it clear that the sum

$$1\times 14+1\times 5+2\times 2+5\times 1+14\times 1$$

is relevant to the problem.

c) In how many ways can one place parentheses around seven dots? Eight dots? Nine dots?

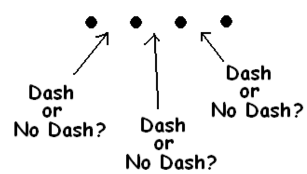
PUZZLE THREE:
Parentheses within parentheses are so hard to read. Let's avoid them!

2	• (•)
5	•• (•)• •(•) (•)(•) (••)
13	••• (•)•• •(•)• ••(•) (••)• •(••) (•••) (•)(•)• (•)•(•) •(•)(•) (•)(••) (••)(•) (•)(•)(•)

etc.

In how many ways can one arrange parentheses around some or all of N dots avoiding nested parentheses? (Worry not this time about having two terms within each set of parentheses.)

Puzzle One and the Powers of Two:
In constructing an arrangement of dashes we have, for each space between a pair of dots, two choices: to insert a dash or not to insert a dash.



With N dots in a row there are $N-1$ spaces and so $N-1$ choices to make. Hence there are 2^{N-1} possibilities for dashes among N dots and we see the powers of two.

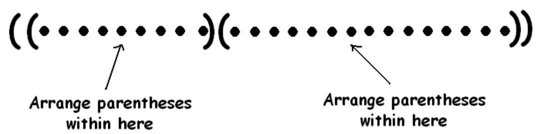
Powers of Three? Suppose we curve the dashes and make them parentheses. Ignoring conventions of matching parentheses, we see that there are 3 options for inserting parentheses between two dots and 9 between three dots:

3	•• •(• •)•
9	••• •)•• •(•• ••(• ••)• •(•(• •)•(• •(•)• •)•)•

Is it clear that for N dots there are 3^{N-1} ways to insert parentheses?

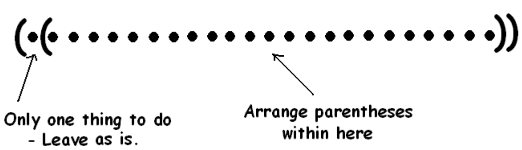
Question: How many of the 3^{N-1} possibilities actually correspond to meaningful arrangements of parentheses?

Puzzle Two and the Parentheses Numbers: In any set of parentheses around a row of dots, the outer most parentheses split the row into two pieces. Each piece is its own independent version of the parentheses problem.



Thus if there are a ways to arrange parentheses among the left portion of dots and b ways among the right portion, there are $a \times b$ arrangements of parentheses on all the dots with this basic left portion/right portion structure.

There is one extra case to consider: one of the "portions" might consist of a single dot.

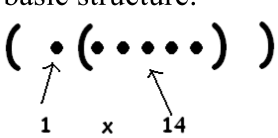


Let's declare that there is 1 way to arrange parentheses around a single dot (leave it alone, I suppose). In this way we can still say "a options for the left portion and b options for the right gives $a \times b$ options in all."

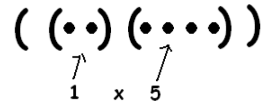
This observation allows us to compute parentheses numbers. So far we have:

One dot = 1 "option"
Two dots = 1 option
Three dots = 2 options
Four dots = 5 options
Five dots = 14 options

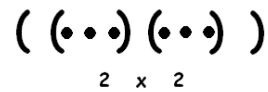
For six dots we see there are $1 \times 14 = 14$ arrangements of parentheses with the following basic structure:

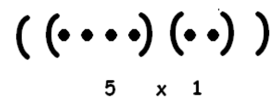


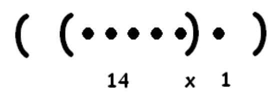
and $1 \times 5 = 5$ arrangements with the structure:



And continuing on …







we see there are

$$\begin{matrix}1\\ \times\\ 14\end{matrix} + \begin{matrix}1\\ \times\\ 5\end{matrix} + \begin{matrix}2\\ \times\\ 2\end{matrix} + \begin{matrix}5\\ \times\\ 1\end{matrix} + \begin{matrix}14\\ \times\\ 1\end{matrix} = 42$$

ways to arrange parentheses around six dots.

In the same manner we have

$$\begin{matrix}1\\ \times\\ 42\end{matrix} + \begin{matrix}1\\ \times\\ 14\end{matrix} + \begin{matrix}2\\ \times\\ 5\end{matrix} + \begin{matrix}5\\ \times\\ 2\end{matrix} + \begin{matrix}14\\ \times\\ 1\end{matrix} + \begin{matrix}42\\ \times\\ 1\end{matrix} = 132$$

arrangements for seven dots. And a general method for computing these numbers is clear: *Write the terms of the sequence established so far forwards and backwards, multiply matching pairs, and sum. This produces the next term.*

> INTERNET RESEARCH: The sequence 1,1,2,5,14,42,132,429, 1430, 4862, … is famous and appears in many disparate and surprising contexts. It was first discovered by Leonhard Euler (1707-83) in connection to counting diagonals in polygons, but was named after Eugene Catalan (1814-94) after he rediscovered them in the context of counting parentheses. The internet reveals many wonders of the Catalan numbers.

Comment: There is a formula for the n th Catalan number. It is $\frac{(2n)!}{(n+1)!n!}$.

See the short video
www.jamestanton.com/?p=1079
for an accessible derivation.

Puzzle Three and the Fibonacci Numbers:
You may be familiar with the Fibonacci numbers

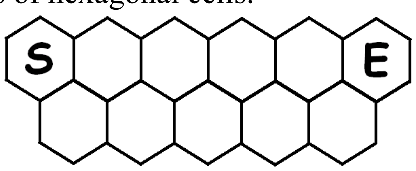
1,1,2,3,5,8,13,21,34,55,89,144,…

They have a rich and surprisingly deep mathematical history. (Again, see the internet!) The sequence begins with a pair of 1s, and thereafter each number in the sequence is the sum of its two predecessors.

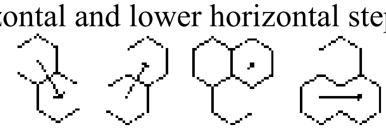
Curious Question: 1 and 144 are square numbers. Is there another square Fibonacci number later in the list?

My personal favourite appearance of the Fibonacci numbers comes from a path-counting puzzle. (To see why I like this so, watch: SURPRISING FIBONACCI APPEARANCES at www.jamestanton.com/?p=959)

HONEYCOMB WALKS
A bee, starting at cell marked S wants to move to the cell marked E in a honeycomb design consisting of two rows of hexagonal cells.



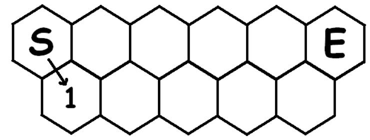
Each step must move to the right and to a neighbouring cell. This gives four possible types of steps: downward diagonal, upward diagonal, high horizontal and lower horizontal steps.



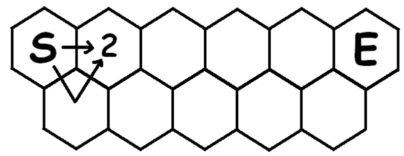
How many different routes are there from S to E using these four types of moves?

As with any complex problem it is often easier to start by examining smaller cases.

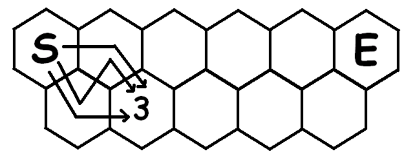
Clearly there is only one route from S to the cell just below it: Take a single downward step.



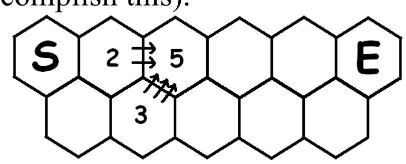
There are two routes from S to the cell directly to its right.



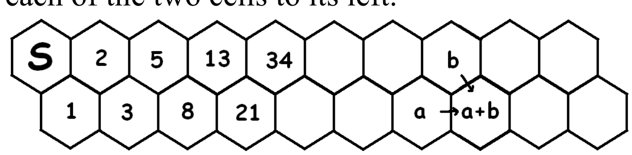
One can check that there are three routes to the next cell along the honeycomb.



To reach the third cell along the top row one has two options: either reach the cell horizontally to its left and then take a horizontal step to the right (there are two ways to accomplish this) OR reach the cell diagonally to its left and then take a diagonal step (and there are three ways to accomplish this).



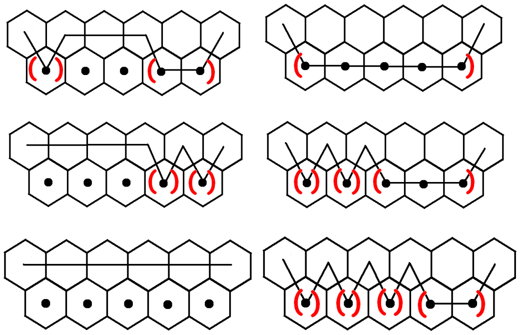
In general, the number ways to reach any particular cell of the honeycomb must be the sum of the two counts of paths to each of the two cells to its left.



This is precisely the construction of the Fibonacci numbers.

Notice that the numbers along the top row of the honeycomb, the counts of paths that end on a top cell, match the sequence of numbers from puzzle three: 2, 5, 13, 34, 89, ….

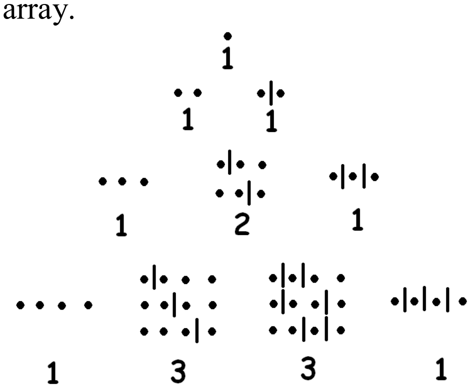
Wordless pictures explain all!



Comment: Looking for a formula for the Fibonacci numbers? See www.jamestanton.com/?p=602 .

PASCAL'S TRIANGLE IS ALL!

Let's go back to puzzle one and arrange all the diagrams that arise there in an array.



The N th row consists of all the dot/dash diagrams containing N dots grouped according to counts of dashes. If we count the number of diagrams in each group Pascal's famous triangle appears!

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
etc.

Comment: A diagram with N dots has $N-1$ spaces, some filled with dashes and some left blank. Thus each diagram can be encoded as a "word" of length $N-1$ with the letters D for dash and B for blank.

• • • = BB   •|• • = DB   • •|• = BD   •|•|• = DD

It is natural to call the very top row of Pascal's triangle the 0 th row since we are really working with $N-1$ rather than N. Then each entry of the array simply counts the number of words with a Bs and b Ds where $a+b$ adds to the row number. This gives the formula $\frac{(a+b)!}{a!b!}$ for each entry of the array, which is precisely the formula for each entry of Pascal's triangle. (Yet another video www.jamestanton.com/?p=656 explains all this in accessible detail.)

***

The powers of two from puzzle one appear in Pascal's triangle by summing the entries of each of its rows:

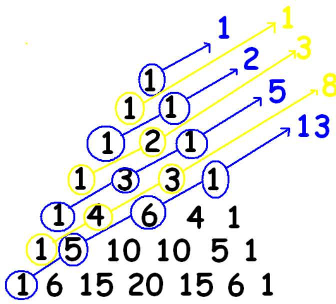
$$1 = 1$$
$$1+1 = 2$$
$$1+2+1 = 4$$
$$1+3+3+1 = 8$$
$$1+4+6+4+1 = 16$$
$$\vdots$$

The parentheses numbers from puzzle two (Catalan numbers) appear in disguise. They are the middle entries of every other row divided by successive counting numbers:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
56 70 56

$$1 \div 1 = 1 \qquad 2 \div 2 = 1 \qquad 6 \div 3 = 2$$
$$20 \div 4 = 5 \qquad 70 \div 5 = 14$$

The Fibonacci numbers of puzzle three appear by adding terms of slanted diagonals:



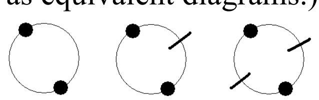
CHALLENGE: Explain why the powers of two, the Catalan numbers and the Fibonacci numbers appear this way. (Can you also find the powers of three in Pascal's triangle?)

Comment: See Chapter 17 of *THINKING MATHEMATICS! Vol 2: Advanced Counting and Advanced Number Systems* available at www.lulu.com for a complete and gentle study of Pascal's triangle.

RESEARCH CORNER:
A Ring Triangle

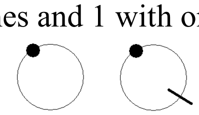
Pascal's triangle arises from inserting dashes between dots arranged in a row. Let's try inserting dashes between dots arranged in a ring.

Examples: For two dots on a ring (and the two spaces between them) there is 1 diagram with no dashes, 1 with one dash, and 1 with two dashes. (Let's consider rotations and reflections of pictures as equivalent diagrams.)



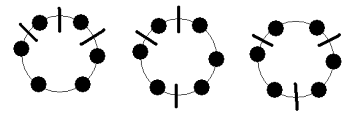
1 1 1

For one dot on a ring, there is 1 diagram with no dashes and 1 with one dash.



1 1

To practice a larger example, there are essentially only three ways to insert three dashes in a diagram of six dots:



3

(Remember: reflections and rotations are considered equivalent.)

Counting the number of ways to insert k dashes into a ring of N dots leads to the following array of numbers:

```
            *
          1   1
        1   1   1
      1   1   1   1
    1   1   2   1   1
  1   1   2   2   1   1
1   1   3   3   3   1   1
           etc.
```

What properties does this array possess? The rows no longer sum to the powers of two. What sequence of numbers comes from summing the row entries?

Is there a version of "ring Catalan numbers" to be found in this array? Is there a version of "ring Fibonacci numbers"?

How does the array change if we don't consider rotations and reflections of diagrams as equivalent? Or how about we only consider rotations of diagrams as equivalent, not reflections?

There is much to be explored and no doubt much to be discovered!

tanton.math@gmail.com