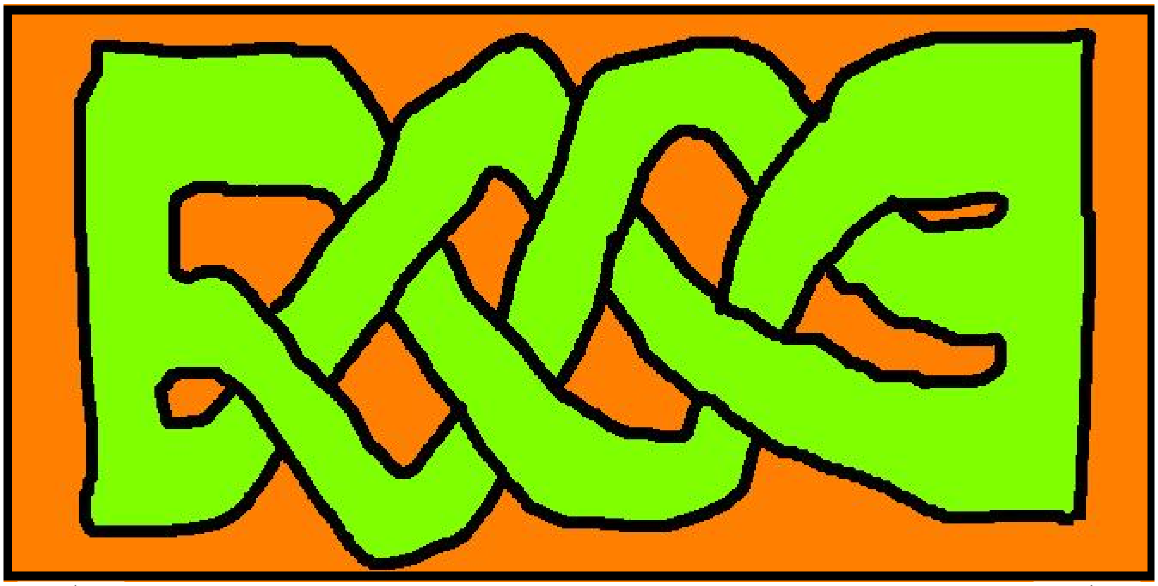

WHAT HO! COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

OCTOBER 2012

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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The MAA has a brand new book. I am excited about it.
(I am so proud of my students!)
www.maa.org/ebooks/crm/MAG.html

"**Mathematics Galore!** showcases some of the best activities and student outcomes of the St. Mark's Institute of Mathematics and invites you to engage in the mathematics yourself! Revel in the delight of deep intellectual play and marvel at the heights to which young scholars can rise. See some great mathematics explained and proved via natural and accessible means.

... Mathematics Galore! offers a large sample of mathematical tidbits and treasures, each immediately enticing, and each a gateway to layers of surprising depth and conundrum. Pick and read essays in no particular order and enjoy the mathematical stories that unfold. Be inspired for your courses, your math clubs and your math circles, or simply enjoy for yourself the bounty of research questions and intriguing puzzlers that lie within."

WARNING: This month's essay is a meaty one. Some heavy reading ahead! (Take it slowly.)

SUMS OF CONSECUTIVE INTEGERS PUZZLE:
Some numbers are the sum of two or more consecutive counting numbers. For example, $15 = 4 + 5 + 6$ and $100 = 18 + 19 + 20 + 21 + 22$.
Some numbers seem to be resistant to being expressed this way. For example, 4 does not equal a sum of two or more consecutive positive integers.

a) Describe all those obstinate numbers. That is, completely classify those numbers which fail to be a sum of two or more consecutive counting numbers.
b) Which numbers fail to be the sum of <u>three</u> or more consecutive counting numbers?

This puzzle and its answers appear in the video www.jamestanton.com/?p=**844** .

ON SUMS OF CONSECUTIVE COUNTING NUMBERS:
Every now and then an email is bandied about the community of math enthusiasts detailing some beautiful patterns in arithmetic. The following example seems to be a recurring favourite:

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

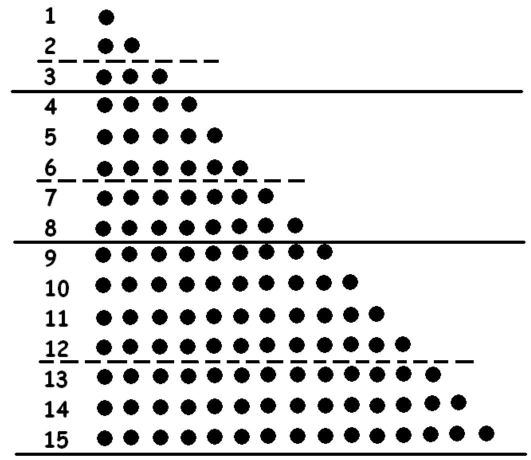
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

$$16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$$

etc.

My reaction to first seeing this example was: "I don't' get it!" Of course I understood the pattern being presented and was clear on how to extend it so as to, supposedly, partition the entire set of counting numbers into balanced sums. But I had no sense of what is really going on and why these sums should continue to hold true. Could the first four, ten, one hundred lines of this be coincidental?

I am a visual mathematician. I see this pattern as a picture of dots:



The claim is that the infinite triangle of dots is broken into "blocks" (given by the solid lines) with each block composed of two parts (each separated by a dashed line) possessing an equal number of dots in each part.

My two basic questions were:

1: **When does a block happen?**
2. **When a block does happen, is it sure to keep happening?**

Answering these questions (and pushing beyond them) is the topic of this month's heady math letter.
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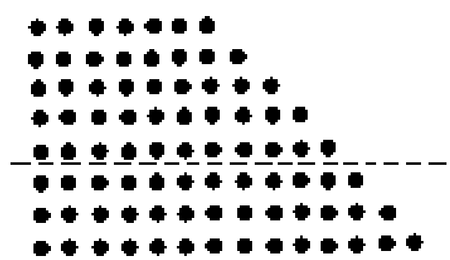
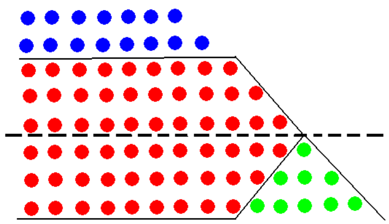
Qu 1: When does a block happen?
Suppose we have a block of dots that works the way we hope, with a top portion of dots equal in number to the bottom set of dots.



[WARNING: Don't take this picture literally: the numbers of dots here actually don't work! I just wish to work with the shape of the block.]

Since the lines in the triangle grow larger as we move downward, the top portion of dots must contain more rows than the bottom potion if they are to equal in counts of dots.

Colour the dots as shown:



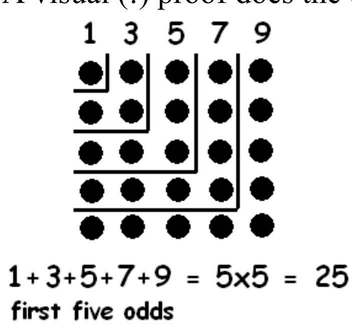
We have the "equation:"

$$blue + red = red + green$$

which shows that the number of blue dots must equal the number of green. And the green dots represent a sum of odd numbers: $1+3+5+\ldots$. That is exciting!

FACT: *The sum of the first* r *odd numbers is* r^2.

Proof: A visual (!) proof does the trick:

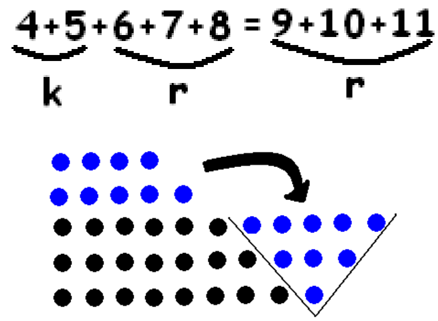


So we see: **Whenever a set of rows in the triangle add to a square number of dots** (the blue dots) **we can create a block that works** (shift the blue dots to the green positions).

In more algebraic terms:

Whenever k consecutive integers add to a square number r^2, the sum of those k integers plus the next r integers equals the sum of the r consecutive integers that follow those!

For example: $4+5=3^2$ has $k=2$ with $r=3$. We have:



Notice that this particular example of a block does not appear in the original pattern given at beginning of this essay!

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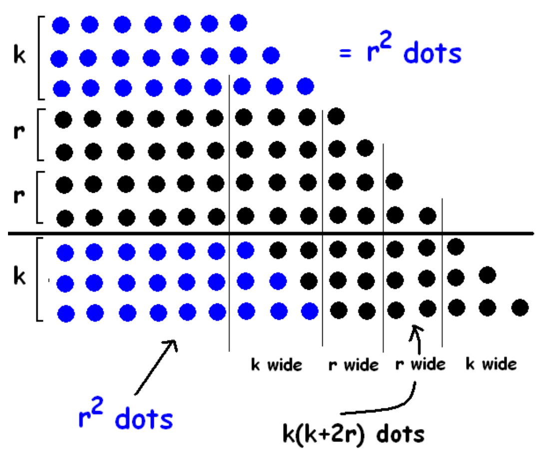
Qu 2: Do blocks keep happening?

The answer is YES!

If k consecutive integers add to a square number r^2, then the set of k numbers after $k+r+r$ terms add to another square number (and so give another block).

Proof:



This diagram shows that if k rows add to r^2, then the next set of k rows after the block add to

$$r^2 + k(k+2r) = r^2 + 2rk + k^2 = (r+k)^2$$

- another square number. [It is a bigger square number this time and so it will produce its own bigger block with k rows plus another $r+k$ rows adding to the next $r+k$ rows.]

For example, from $4+5=3^2$ we obtain

$$(4+5)+(6+7+8)=(9+10+11)$$

which then gives

$$(12+13)+(14+\cdots+18)=(19+\cdots+23)$$

and this yields

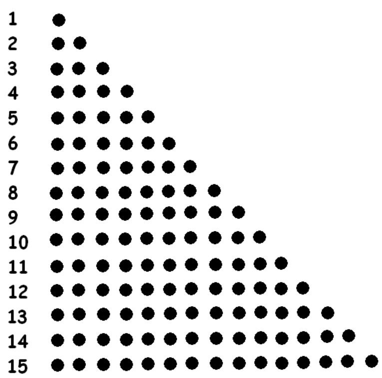
$$(24+25)+(26+\cdots+32)=(33+\cdots+39)$$

and

$$(40+41)+(42+\cdots+50)=(51+\cdots+59)$$

and so on, with each pair of middle and final sets of parentheses in one equation containing $k=2$ more terms than for the previous equation.

Once we have one block we now know that the counting numbers thereafter partition perfectly into blocks. And to find one block we need only find some consecutive counting numbers that sum to a square number.



The very first row of this triangle has a square number of dots: $1=1^2$ (Here $k=1$ and $r=1$.) This gives:

$$(1)+(2)=(3)$$
$$(4)+(5+6)=(7+8)$$
$$(9)+(10+11+12)=(13+14+15)$$

etc.

which is the example at the beginning of this essay.

The first 8 rows of the triangle have a square number of dots: $1+2+\cdots+8=36=6^2$. (Here $k=8$ and $r=6$.) This gives the partition:

$$(1+\cdots+8)+(9+\cdots+14)=(15+\cdots+20)$$
$$(21+\cdots+28)+(29+\cdots+36)=(37+\cdots+44)$$
$$(45+\cdots+50)+(51+\cdots+58)=(59+\cdots+66)$$

etc.

The first 49 rows of the triangle have a square number of dots: $1+2+\cdots+49=1225=35^2$. (Here $k=49$ and $r=35$.) This generates a new partition of the counting numbers.

The first 288 rows of the triangle have a square number of dots: $1+2+\cdots+288=204^2$. (Here $k=288$ and $r=204$.) This generates a new partition of the counting numbers.

COMMENT: See the essay on "squangular numbers" to see how I am coming up with these crazy examples! www.jamestanton.com/?p=**519** .

From the example $4+5=3^2$ which generates

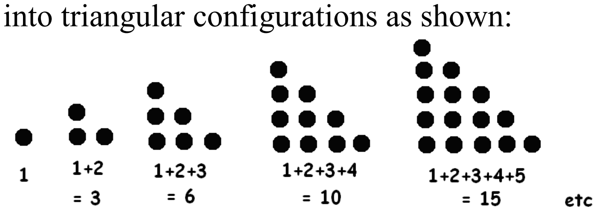
$$(4+5)+(6+7+8)=(9+10+11)$$
$$(12+13)+(14+\cdots+18)=(19+\cdots+23)$$
$$(24+25)+(26+\cdots+32)=(33+\cdots+39)$$
$$(40+41)+(42+\cdots+50)=(51+\cdots+59)$$

etc.

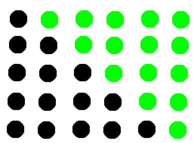
we can insert $1+2=3$ at its beginning to obtain yet another partition of the counting numbers. And I am sure there are plenty more!

HOW TO FIND MORE!

The triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, … are the numbers that arise from arranging dots into triangular configurations as shown:



Let $T_n = 1+2+3+\cdots+n$ denote the nth triangle number. The following diagram shows that two copies of T_n form an $n\times(n+1)$ rectangle, and so $T_n = \frac{1}{2}n(n+1)$. This proves the famous formula: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.



Any sum of consecutive integers is a difference of two triangle numbers. For example, $4+5+6$ can be written:

$$(1+2+3+4+5+6)-(1+2+3)$$
$$= T_6 - T_3$$

and in a block we have one sum of consecutive integers equally a next. For example we see that $4+5+6=7+8$ reads:

$$T_6 - T_4 = T_8 - T_6$$

Algebra gives:

$$\frac{T_4+T_8}{2} = T_6$$

Thus each block in a partition of the counting numbers gives two triangle numbers whose average is another triangle number!

The partition of the counting numbers given at the start of this essay is:

$$T_2 - T_0 = T_3 - T_2$$
$$T_6 - T_3 = T_8 - T_6$$
$$T_{12} - T_8 = T_{15} - T_{12}$$
$$T_{20} - T_{15} = T_{24} - T_{20}$$

etc.

(set $T_0 = 0$) or equivalently it is the sequence of averages: $T_2 = ave(T_0, T_3)$, $T_6 = ave(T_3, T_8)$, $T_{12} = ave(T_8, T_{15}), \ldots$

[In general $T_{p(p+1)} = ave(T_{p^2-1}, T_{(p+1)^2-1})$.]

And conversely: If we happen to find a triangle number that is the average of two other triangular numbers, $T_{17} = 153$ is the average of $T_3 = 6$ and $T_{24} = 300$, for instance, then we've found a new block in counting numbers:

$$T_{17} = \frac{T_3 + T_{24}}{2}$$

can be rewritten $T_{17} - T_3 = T_{24} - T_{17}$, which states:

$$(4+5+\cdots+10)+(11+\cdots+17) = (18+19+\cdots+24)$$

and this is a block. (Here: 7 terms + 7 terms $= 7$ terms, so $k=7$, $r=7$.)

How do we find triangle numbers that are the averages of other triangle numbers?

We are looking for values a, b and c so that $T_c = \frac{T_a + T_b}{2}$, that is so that:

$$\frac{c(c+1)}{2} = \frac{1}{2}\left(\frac{a(a+1)}{2} + \frac{b(b+1)}{2}\right).$$

Algebra shows that this is equivalent to equation:

$$(2c+1)^2 = (a+b+1)^2 + (a-b)^2$$

which is an equation of the form:

$$z^2 = x^2 + y^2$$

All we need find are examples of Pythagorean triples – and there are plenty of those! (See the video www.jamestanton.com/?p=628 for a good way to hunt for them.)

From $13^2 = 5^2 + 12^2$, for example, we obtain $c=6$, $a=8$, $b=3$ and

$$T_6 = 21 = \frac{T_3 + T_8}{2} = \frac{6+36}{2}.$$

From $25^2 = 7^2 + 24^2$ we have $c=12$, $a=15$, $b=8$ and

$$T_{12} = 78 = \frac{T_8 + T_{15}}{2} = \frac{36+120}{2}.$$

ALGEBRA GALORE:
Suppose we have an example of three integers satisfying:

$$x^2 + y^2 = z^2 \text{, with } z \text{ odd.}$$

Put

$$a = \frac{x+y-1}{2}$$

$$b = \frac{x-y-1}{2}$$

$$c = \frac{z-1}{2}$$

Then we have a solution to:

$$T_c = \frac{T_a + T_b}{2}$$

And conversely, if given the values a, b, c first, set $x = a+b+1$, $y = a-b$, $z = 2c+1$ to get the Pythagorean triple.

If instead, from the Pythagorean triple, we set:

$$A = x + y$$

$$B = x - y$$

$$C = z$$

we get:

$$\frac{A^2 + B^2}{2} = C^2 \text{, with } C \text{ odd.}$$

This is an example of two square numbers whose average is again square. WHOA!

Conversely, if one has an example of squares that relate this way, set $x = \frac{A+B}{2}$, $y = \frac{A-B}{2}$, $z = C$ to return to the Pythagorean triple.

EXERCISE: Check all this algebra!

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IN SUMMARY: To find a block of consecutive counting numbers that work in the manner set by the example at the start of this essay either:

Hunt for k consecutive integers that sum to a square number

OR

Hunt for two triangle numbers whose average is again triangular.

OR

Hunt for two square numbers whose average is an odd square.

OR

Hunt for a Pythagorean triple with largest term odd.

These are all equivalent actions!

And once we have found one block that works, the counting numbers will, for sure, continue to keep partitioning this way from that point forward!

Exercise: Find infinitely many different ways to partition the counting numbers into blocks. (And be sure to hit "reply all" with one of your examples if you ever receive a "beautiful arithmetic" email!)

RESEARCH CORNER:
How much of this work applies to other arithmetic sequences? For example, the sequence $3, 5, 7, 9, ...$ partitions as:

$$3 + 5 + 7 + 9 = 11 + 13$$

$$15 + 17 + \cdots + 25 = 27 + \cdots + 33$$

etc.

(There is no full partition for $1, 3, 5, 7, 9, \ldots$. Can you see why?)

Explore when a sequence of the form

$$a, a+d, a+2d, a+3d, a+4d, ...$$

partitions into blocks and find connections to Pythagorean triples and averages of special numbers!