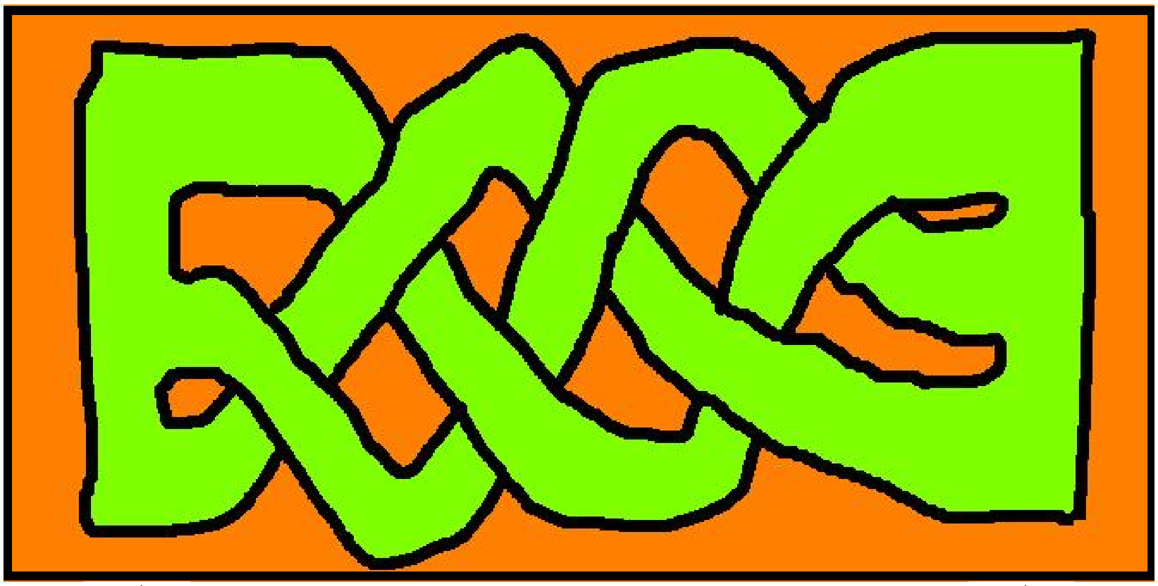


WAY COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

The second of a new Math Letter series!

MAY 2012

Welcome to the second Cool Math Letter from the nation's capital. Lots of fun for students, mathematicians, teachers, and general math enthusiasts. (How many sets of people is that?)

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing, I would be delighted to mention it here.*

SUMMER INSTITUTE: The Boston Math Circle, www.themathcircle.org, holds its 5th annual Math Circle Teacher Training Institute July 8–14 on the Notre Dame campus in South Bend, Indiana. Innovative, refreshing, eye-opening and spectacular! Applications welcome.

MATHPICKLE: Check out Gordon Hamilton's truly fabulous www.mathpickle.com. Curricular puzzle books, wonderful videos, and million dollar questions each worth – you guessed it – a million $$s!

COOL BOOK: Robin Padron's *HOW TO HOMESCHOOL MATH-Even if you hate fractions* is so worth the read, even for teachers! An illuminating perspective on the K-12 curriculum we teach.

A PUZZLER: *Here's a classic.*

PLAYING WITH NUMBERS:

Write the numbers 1 though 10 on the board. Pick any two numbers, erase them, and replace them with the single number given by their sum plus their product. (So, if you choose to erase the numbers a and b, replace them with $a+b+ab$.) You now have nine numbers on the board.

Do this again: Pick any two numbers, erase them, and replace them with their sum plus product. You now have eight numbers on the board.

Do this seven more times until you have a single number on the board.

Why do all who play this game end up with the same single number at the end? What is that final number?

Perhaps try playing this game several times for yourself first with the numbers 1, 2, 3, 4. Do you indeed get the same final answer each and every time? Any thoughts as to why?

If you are game, here are some not-so-classic wild variations!

VARIATION 1: Replace a pair of numbers a and b with $a+b+1$ instead. Again, why must the same single final answer appear each and every this variation is played?

VARIATION 2 for those who don't mind fractions: This time replace a pair of numbers a and b with the fraction $\frac{ab}{a+b}$. You'll have fractions within fractions as you play, but the final answer will again be constant!

VARIATION 3 for those who don't mind square roots: Replace a and b with:

$$\sqrt{a^2+b^2}.$$

Same final answer for sure!

VARIATION 4 for those who don't mind logarithms: Replace a and b with:

$$\log\left(10^a+10^b\right).$$

Same end number every time!

VARIATION 5 for those who don't mind big numbers: Replace a and b with:

$$(a+88)(b+88)-88.$$

VARIATION 6 for those who like common factors: Replace a and b with $\gcd(a,b)$, the greatest common factor (divisor) of a and b.

BETTER YET … Replace a and b with:

$$\gcd(a+1,b+1)-1.$$

[$\gcd(0,N)=N$, by the way.]

VARIATION 7 for those who like complicated things: Replace a and b with:

$$\frac{ab}{\sqrt{a^2+b^2}}.$$

OR with:

$$\left(1+\sqrt{a}+\sqrt{b}\right)^2$$

OR with:

$$\sqrt[3]{a^3+a^3b^3+b^3}$$

OR with …

What is going on? How can one come up with these crazy formulas? Why do they all lead to invariant final numbers?

Read on!

ADDING AND ORDER:
Folk often say that the order in which one sums a list of numbers does not matter. The phrase "the order in which" actually means two things here:

MEANING ONE: If given two numbers a and b, one can either add a to b or one can add b to a. The results are the same. This is usually written as:

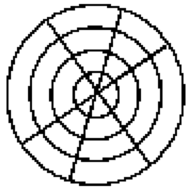
$$a+b=b+a$$

(and for those that like jargon, it is called the commutative property of addition).

We like to believe that addition is commutative as it seems to be patently true at the level of basic counting. For example, just look at this picture both forwards and backwards to see that $2+3$ and $3+2$ must be the same:

Question: We also like to believe that multiplication is commutative: $a\times b=b\times a$. Is this obviously true? At the level of basic counting folk often say that "multiplication is repeated addition." For example, 4×3 represents "four groups of three," $3+3+3+3$. Is it at all clear that 117 groups of 863 is the same as 863 groups of 117?

CHALLENGE: Here's a new operation on counting numbers called "circle-ation." To compute $4\circ 3$, draw four concentric circles and three diameters. Then $4\circ 3=24$, the number of pieces that result.



Is it obvious that circle-ation is commutative, that $a\circ b=b\circ a$ always?

MEANING TWO: Without changing the order in which numbers appear in the list, it does not matter which consecutive pairs one adds in turn. For example, in

$$2+3+4$$

we can compute $2+3$ first to get 5, and then compute $5+4=9$. Alternatively, we can add first $3+4$ to get 7, and then compute $2+7=9$.

People call this property the associative rule and write it as:

$$a+\left(b+c\right)=\left(a+b\right)+c$$

There are more complicated variations of this rule, such as:

$$a+\left(\left(b+\left(c+d\right)\right)+e\right)=\left(\left(a+b\right)+c\right)+\left(d+e\right)$$

but all these complicated variations boil down to relying on the one very basic rule described above.

HARD CHALLENGE: Figure out how to properly state this claim ... and then prove it!

We like to believe that addition – and multiplication – in the realm of real numbers satisfy the associative rule.

CHALLENGE: Think of a mathematical operation that does not obey the associative law. Can you think of a <u>commutative</u> mathematical operation that is not associative?

WEIRD EXAMPLE: Given two positive numbers a and b define $a*b=a^{\log b}$. Prove that $a*b=b*a$! Is this operation associative?

EXPLAINING THE PUZZLER:

If it does not matter in which order one combines numbers from a pair, nor which pairs one chooses to combine in some order, then combining any set of ten numbers two numbers at a time must lead to the same final result!

That is, if an operation $a \square b$ between two numbers a and b is commutative and associative, then the manner in which one computes

$$1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 \square 10$$

for the numbers 1 through 10 does not matter – all choices will lead to the same final answer.

EXAMPLE: Suppose the puzzler read: *For the numbers 1 through 10 on the board, erase any two numbers a and b and replace them with their sum $a+b$. Repeat until a single answer is produced.* Can you truly see for yourself that the final answer simply must be $1+2+3+4+5+6+7+8+9+10$? If instead each time we replaced a and b with their product $a \cdot b$, then the final answer would have to be …?

The classic version of the puzzle uses the operation:

$$a * b = a + b + ab$$

This is commutative: $a * b = b * a$, and a little work shows it is also associative:

$$\begin{aligned}(a*b)*c &= (a+b+ab)*c \\ &= a+b+ab+c+(a+b+ab)c \\ &= a+b+c+ab+ac+bc+abc\end{aligned}$$

$$\begin{aligned}a*(b*c) &= a*(b+c+bc) \\ &= a+b+c+bc+a(b+c+bc) \\ &= a+b+c+ab+ac+bc+abc\end{aligned}$$

So $(a*b)*c = a*(b*c)$.

If we are extra clever we may notice that $a*b = a+b+ab = (a+1)(b+1)-1$ and $a*b*c = (a+1)(b+1)(c+1)-1$ and so on. The final number in the puzzler is:

$$\begin{aligned}1*2*3*\cdots*10 &= 2 \cdot 3 \cdot \cdots \cdot 11 - 1 \\ &= 11! - 1 \\ &= 39916799\end{aligned}$$

All the variations to the puzzler are alternative operations between two numbers that are commutative and associative. But I am not a clever person and I didn't do anything hard to invent them. I simply noted:

If $a*b$ is a commutative and associative operation on numbers, and F is any function that takes numbers to numbers with a well-defined backwards version F^{-1}, then

$$a \square b = F^{-1}\left(F(a) * F(b)\right)$$

is a new commutative and associative operation.

We have three examples of what to use for $*$ to start with:

$$a*b = a+b$$

$$a*b = ab$$

and

$$a*b = a+b+ab = (a+1)(b+1)-1.$$

F can be any function that can be undone. For example, F could be "squaring" in which case F^{-1} is "square rooting," or F could be "use as exponents" with F^{-1} "logarithms," or F could be "adding 88" with F^{-1} "subtracting 88."

Before launching into the examples, let's run through a proof of why the observation works. (And then explain – slowly - how to use it!)

Proof: (Warning! ICKY TO READ!)
Do write $a\square b$ for $F^{-1}\left(F(a)*F(b)\right)$.
Now

$$\begin{aligned} b\square a &= F^{-1}\left(F(b)*F(a)\right) \\ &= F^{-1}\left(F(a)*F(b)\right) \\ &= a\square b \end{aligned}$$

using the fact that * is commutative.

$$\begin{aligned} (a\square b)\square c &= F^{-1}\left(F(a\square b)*F(c)\right) \\ &= F^{-1}\left(F\left(F^{-1}\left(F(a)*F(b)\right)\right)*F(c)\right) \\ &= F^{-1}\left(\left(F(a)*F(b)\right)*F(c)\right) \end{aligned}$$

$$\begin{aligned} a\square(b\square c) &= F^{-1}\left(F(a)*F(b\square c)\right) \\ &= F^{-1}\left(F(a)*F\left(F^{-1}\left(F(b)*F(c)\right)\right)\right) \\ &= F^{-1}\left(F(a)*\left(F(b)*F(c)\right)\right) \end{aligned}$$

and these are equal since * is associative. DONE!

Here's how to read $F^{-1}\left(F(a)*F(b)\right)$:

Apply F to your two numbers a and b, and combine the results in the way you are comfortable with. Then hit the overall result with that which undoes F.

For example, let's use addition with F being squaring. Hitting a and b with F gives a^2 and b^2. Combining these results via addition produces a^2+b^2. Finally applying the undoing of F to the overall result gives $\sqrt{a^2+b^2}$.
Thus $a\square b=\sqrt{a^2+b^2}$ is a new commutative and associative operation! We have variation 3.

Let F be "add one" and let * be addition. Then

$$\begin{aligned} a\square b &= (a+1)+(b+1)-1 \\ &= a+b+1 \end{aligned}$$

which is variation 1.

If we use the same F combined with multiplication instead we get:

$$\begin{aligned} a\square b &= (a+1)(b+1)-1 \\ &= a+b+ab \end{aligned}$$

- the operation of the puzzler!

Let $F(x)=10^x$. Then F^{-1}, the undoing of F is $\log x$. This, used with addition, gives variation 4.

Let $F(x)=\frac{1}{x}$. Then $F^{-1}(x)=\frac{1}{x}$ as well. (Inversion undoes itself!) Applying this with addition gives:

$$a\square b=\frac{1}{\frac{1}{a}+\frac{1}{b}}=\frac{ab}{a+b}$$

(So here we inverted a and b, added the results, and inverted the overall result.)

Of course, once we've created new commutative and associative operations we can use them as the basis for even more!

For example, using $a\square b=a+b+ab$ with F being squaring produces $a\triangle b=\sqrt{a^2+b^2+a^2b^2}$ as a new operation to use.

CHALLENGE: Figure out how I created the remaining variations at the beginning of this letter. Find general formulas for $a\square b\square c$ and $a\square b\square c\square d$ for each of them. What do you notice? **PERSONAL RESEARCH:** If someone hands you a commutative and associative operation that works on all real numbers, must it come from addition or from multiplication via some function F as described in this letter?

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