



# CURIOUS MATHEMATICS FOR FUN AND JOY

### **MARCH 2013**

#### PROMOTIONAL CORNER:

Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.

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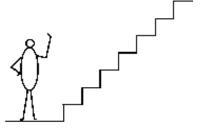
A *math circle*, be it for students or for teachers, is a tremendous experience for one and all who enjoy lively, engaging, rich, deep and meaningful mathematical exploration. There are MANY of them across the nation. Check out:

sigmaa.maa.org/mcst, www.mathcircles.org, and www.mathteacherscircle.org

to learn more about them (and to find one near you!)

#### 

A CLASSIC PUZZLER: I am standing at the base of a set of stairs.



In how many different ways can I walk 10 steps up and 10 steps down?

## THE PARENTHESES NUMBERS

The following puzzler appeared in this month's Curriculum essay (though it was presented there in terms of vinculums).

#### **COUNTING PARENTHESES:**

Consider the sum 1+2+3.

There are **two** ways to place parentheses in this sum (keeping the order of the terms the same) so that one is summing only two numbers at a time:

$$(1+2)+3=3+3=6$$

$$1+(2+3)=1+5=6$$

Consider the sum 1+2+3+4.

There are **five** ways to insert parentheses so that only two numbers are ever being added at a time:

$$((1+2)+3)+4=(3+3)+4=6+4=10$$
  
 $(1+(2+3))+4=(1+5)+4=6+4=10$ 

$$1+((2+3)+4)=1+(5+4)=1+9=10$$

$$1 + (2 + (3 + 4)) = 1 + (2 + 7) = 1 + 9 = 10$$

$$(1+2)+(3+4)=3+7=10$$

[Notice the pair of "simultaneous" parentheses in the last line.]

a) There are **fourteen** ways to insert parentheses into the sum 1+2+3+4+5. List all 14 ways!

There is really only **one** way to place parentheses over the sum 1+2. (Though we don't normally do it!)

So far we have the "parentheses numbers":

### 1 2 5 14

b) Begin listing all the ways to place parentheses throughout a sum of six numbers. Do this in a systematic manner and see if you can explain why the count of possibilities is:

$$1 \times 14 + 1 \times 5 + 2 \times 2 + 5 \times 1 + 14 \times 1 = 42$$
.

- c) What is the next parentheses number after this? And the next?
- d) Is there an explicit formula for the *n* th parentheses number?

#### 

Just so that we don't confuse numbers and symbols, let's place parentheses around sums of symbols.

There is **ONE** way to place parentheses in the expression a + b:

$$(a+b)$$

**TWO** ways for the expression a + b + c:

$$(a+b)+c$$

$$a+(b+c)$$

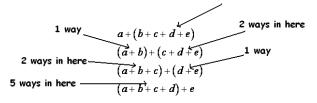
[To match the first answer, we can imagine there is an outermost set of parentheses in these and all the answers that follow:

$$((a+b)+c)$$
 and  $(a+(b+c))$ .]

**FIVE** ways for the expression a+b+c+d:

$$((a+b)+c)+d a+((b+c)+d)$$
$$(a+(b+c))+d a+(b+(c+d))$$
$$(a+b)+(c+d)$$

**FOURTEEN** ways for a+b+c+d+e:



This is a total of  $5+1\times 2+2\times 1+5=14$  arrangements.

**Comment:** This diagram suggests it would be convenient to deem the first parentheses number (the number for just one symbol "a") to be 1. The sum of fourteen arrangements is then expressed as:  $1 \times 5 + 1 \times 2 + 2 \times 1 + 5 \times 1 = 14$ .

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Here is the structure for the **FORTY-TWO** arrangements for a sum of six symbols:

1 way 14 ways
$$a + (b+c+d+e+f)$$

1 way 5 ways
 $(a+b)+(c+d+e+f)$ 

2 ways 2 ways
 $(a+b+c)+(d+e+f)$ 

5 ways 1 way
 $(a+b+c+d)+(e+f)$ 

14 ways 1 way
 $(a+b+c+d)+(e+f)$ 

The count is  $1 \times 14 + 1 \times 5 + 2 \times 2 + 5 \times 1 + 14 \times 1 = 42$ .

The following is hopefully now clear:

The list of parentheses  $P_n$  numbers begin 1, 1, 2, 5, 14, 42,... with the n th number in this list representing the number of ways to insert parentheses into a sum of n symbols (with the convention that this count be 1 for one symbol).

Any next number is constructed by writing the list of previous numbers both forwards and backwards, multiplying matching terms, and summing the result. Thus the seventh parentheses numbers, for instance, is 132:

The list of parentheses numbers continues:

### AN EXPLICIT FORMULA

Although we now have the means to compute any number in this list, the method we have for doing it is far from convenient. For example, to compute the 100 th parentheses number our current method requires us to first compute the 99 values

before it. It would be nice if there is an explicit formula for the *n* th parenthesis number. Is there one? YES!

(Comment: Those who have been following my essays for a number of years will recognize the following proof from the March 2011 newsletter of the St. Mark's Institute of Mathematics.)

It turns out to be easier to first solve a looser problem:

In how many ways can one arrange parentheses around a sum of n symbols if the order of the letters may change?

For example, there are now 2 ways to place parentheses around two letters:

(a+b) and (b+a)

and 12 ways around three letters:  

$$(a+b)+c$$
;  $a+(b+c)$   
 $(a+c)+b$ ;  $a+(c+b)$   
 $(b+a)+c$ ;  $b+(a+c)$   
 $(b+c)+a$ ;  $b+(c+a)$   
 $(c+a)+b$ ;  $c+(a+b)$   
 $(c+b)+a$ ;  $c+(b+a)$ 

To obtain the next number in the sequence we need to consider a fourth letter d.

For each of the sums above we could place the d at its beginning and create d + (sum)

OR we could place the d at its end and produce instead (sum) + d.

(So, for example, 
$$(a+b)+c$$
 becomes  $d+((a+b)+c)$  or  $((a+b)+c)+d$ .)

OR we could insert d somewhere within the sum.

Now in each sum we have on three symbols there are two + signs. And each + is part of a sum of two quantities X + Y. For example, in (a + b) + c, the first + has X = a, Y = b and the second + has X = (a + b), Y = c.

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There are four ways to insert d within a sum near a given + sign. We can change X + Y to one of the following:

$$(d+X)+Y$$
  $(X+d)+Y$   
 $X+(d+Y)$   $X+(Y+d)$ 

For instance, from (a+b)+c we obtain:

First + Second +
$$((d+a)+b)+c \qquad (d+(a+b))+c$$

$$((a+d)+b)+c \qquad ((a+b)+d)+c$$

$$(a+(d+b))+c \qquad (a+b)+(d+c)$$

$$(a+(b+d))+c \qquad (a+b)+(c+d)$$

And we still have the first two possibilities: d + ((a+b)+c) and ((a+b)+c)+d

So each of the 12 solutions for a sum of three symbols gives us 10 possibilities for the insertion of the letter d. This gives 120 arrangements of parentheses on a sum of four symbols. Moreover, this is all of them: every four-symbol arrangement arises as one of these. (Think about taking away the letter d.)

In general: Suppose there are  $A_n$  solutions to the n-symbol problem. Each solution has n-1 plus signs. For each + there are four ways to insert a next letter along with the two options of inserting a next letter at the beginning, d + (sum), or at the end, (sum) + d. Thus each solution gives 4(n-1) + 2 = 4n - 2 solutions with one more letter. The number of solutions to the n+1 letter problem is hence:

$$A_{n+1} = \left(4n-2\right)A_n.$$

We have seen two solutions to the two-symbol problem: (a+b) and (b+a). And so  $A_2 = 2$ .

It now follows that:

$$A_3 = (4 \cdot 2 - 2) A_2 = 6 \times 2 = 12$$

$$A_4 = (4.3 - 2) A_3 = 10 \times 6 \times 2 = 120$$

$$A_5 = (4 \cdot 4 - 2) A_4 = 14 \times 10 \times 6 \times 2 = 1680$$
and so on.

In general:

$$A_{n} = 2 \cdot 6 \cdot 10 \cdot 14 \cdot \dots \cdot (4n - 6)$$

$$= 2^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n - 3)$$

$$= 2^{n-1} \frac{(2n - 2)!}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \dots \cdot (2n - 2)}$$

$$= 2^{n-1} \frac{(2n - 2)!}{2^{n-1} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n - 1)}$$

$$= \frac{(2n - 2)!}{(n - 1)!}$$

Now, to find the parentheses numbers ...

 $A_n$  = the number of ways to insert parentheses in a sum of n symbols while allowing a change of order of the symbols.

 $P_n$  = the number of ways to insert parentheses in a sum of n symbols without change to the order of the symbols.

As there are n! ways to arrange n letters, the n th parenthesis number must be:

$$P_n = \frac{A_n}{n!} = \frac{(2n-2)!}{n!(n-1)!}.$$

We have a formula for the n-th parentheses number!

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### SOME HISTORY

In 1848 Belgian mathematician Eugène Charles Catalan asked a question similar to ours:

In how many ways can one arrange n pairs of parentheses?

He assumed the parentheses were "free-floating." For example, there is one way to arrange one pair: ( ), and there are two ways to arrange two pairs: ( ( ) ) and ( ) ( ).

**Comment**: Do you see this is the same challenge as the opening stair-climbing puzzle? As one reads from left to right, the count of closing parentheses ")" can never exceed the count of opening parentheses "(". In our puzzler, in reading from left to right, the count of D steps can never exceed the count of U steps. (One cannot step below the floor.)

Catalan discovered the same sequence of numbers 1, 2, 5, 14, 42,... we have. (And if we declare that there is just 1 way to arrange no parentheses, then his and our sequences are absolutely identical.)

**Comment:** This is not quite true: We seem to be offset by an index of one.

There is **one** way to write down a single pair of parentheses: (). By our convention, there is **one** way to place parentheses around a single symbol a.

There are **two** ways to write down a pair of parentheses: (()) and ()(). There is **one** way to place parentheses around a sum of two symbols: (a+b).

There are **five** ways to write down three parentheses: ((())), (()), (()), (()), (()), and (), (()). There are **two** ways to place parentheses around a sum of three symbols. (a+b)+c and a+(b+c).

And so on.

#### A NOT-OBVIOUS CHALLENGE:

Look at the two answers to Catalan's challenge and our two ways of arranging parentheses:

Two parentheses:	Three symbols
(())	(a+b)+c
()()	a+(b+c)

Is there an obvious correspondence between these solutions?

How about between the five answer options?

Three parentheses:	Four Symbols:
((()))	a+(b+(c+d))
(())()	a+((b+c)+d)
()(())	(a+b)+(c+d)
(()())	(a+(b+c))+d
()()()	((a+b)+c)+d

Is it at all obvious that Catalan's challenge should give the same sequence of numbers (offset by an index of one) as our parentheses numbers?

**HINT**: Place an outer-set of parentheses around each of our sums and look at only the symbols "(" and "+" in what we write.

For example, read 
$$((a+(b+c))+d)$$
 as  $((+(++))+d)$ 

Once we have convinced ourselves that Catalan's numbers are the same as our parentheses numbers (off by an index one) we can declare:

**Definition**: The Catalan numbers  $C_n$  are

$$C_n = P_{n+1} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \frac{(2n)!}{n!n!}.$$

**Comment:** The answer to the opening puzzler is  $C_{10} = P_{11} = 16796$ . Did you find and list all the up/down possibilities?

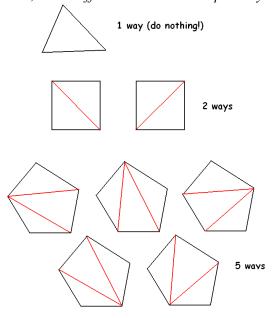
Swiss mathematician Leonhard Euler (1707-1783) was aware of the same sequence of numbers decades earlier through a handshake challenge:

In how many ways can n pairs of people sitting at a circular table reach across and shake hands simultaneously? No pair of hands shaking may cross.

There is 1 way for single pair to shake hands, and 2 ways to four people to shake hands simultaneously. (Diagonal shakes cross and so are not allowed.) Check that there are 5 ways for six people at a table to shake hands simultaneously.

Euler had extreme difficulty in finding a formula for this sequence. He succeeded, but he did not like his complicated approach.

**Challenge:** Actually Euler was led to study this sequence of numbers by a question asked of him in a 1751 letter from Christian Goldbach: *In how many ways can one subdivide a convex n-sided polygon into triangles using only lines drawn from vertex to vertex? The lines drawn may not cross. Also, count different orientations separately.* 



Is the same or a different challenge to Euler's handshake problem? Is it obvious that this problem should be connected to Catalan's parentheses problem?

### <u>,</u>

**RESEARCH CORNER:** 3N people sit at a circular table. In how many different ways can they reach across to hold hands and make triangles? Arms may not cross.

Is there a parentheses version of this challenge?

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