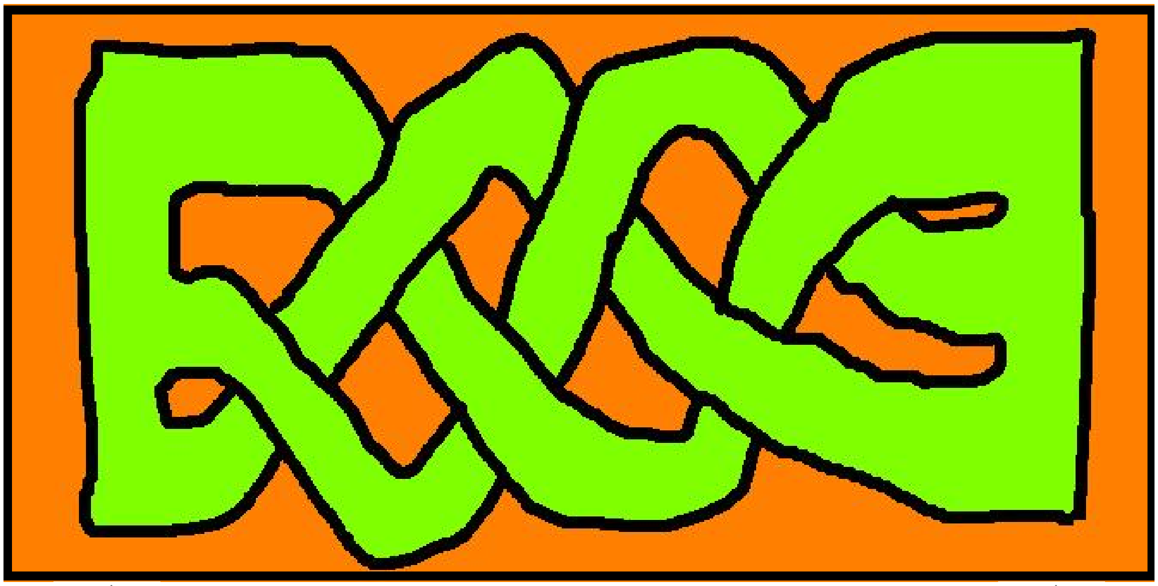


WOW! COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

The Fourth Math Letter

JULY 2012

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

MATH TEACHERS' CIRCLE: Supported by the American Institute of Mathematics, the Math Teacher's Circle group works very hard to support math teachers interested in becoming thoughtful and joyous mathematical thinkers and doers become joyous and thoughtful mathematical thinkers and doers! Lots of workshops and support sessions and resources. Check out www.mathteacherscircle.org.

And while on the Math Circle theme …. **THE NATIONAL ASSOCIATION OF MATH CIRCLES**, www.mathcircles.org, supported by MSRI, is a tremendous resource for those interested in participating in or starting a math circle.

And the book to read on Math Circles, what they and what tremendous good they offer, is:

OUT OF THE LABYRINTH: Setting Mathematics Free

by Robert Kaplan and Ellen Kaplan.

AVERAGE AGE PUZZLE:
Forty students sitting in a circle possess the curious property that the age of any one student in the circle equals the average of the ages of his or her two immediate neighbours. Betty and Jake each are part of the circle. Betty is 9 years old. How old is Jake?

WHICH MEAN DO YOU MEAN?

Here is a question from my video http://www.jamestanton.com/?p=1033

On a recent road-trip I spent the first half of the journey driving at an exact speed of 40 miles an hour and the second half at an exact speed of 65 miles per hour. What was my average speed for the entire trip?

This question is ambiguous: What do I mean by "half the journey:" half the time or half the distance? Each interpretation leads to its own answer!

Half the Time: Suppose I spent T hours driving at 40 mph and T hours at 65 mph. Then I traveled a total distance of $40T+65T$ miles over $2T$ hours. My average speed was

$$\frac{40T+65T}{2T}=\frac{40+65}{2}=52.5 \text{ mph.}$$

Half the Distance: Suppose I traveled X miles at a speed of 40 mph and then X miles at 65 mph. The first part of my journey thus took $X/40$ hours and the second part $X/65$ hours. I traveled a total of $2X$ miles in $\frac{X}{40}+\frac{X}{65}$ hours. My average speed was

$$\frac{2X}{\frac{X}{40}+\frac{X}{65}}=\frac{2}{\frac{1}{40}+\frac{1}{65}}\approx 49.52 \text{ mph.}$$

The first answer involved the *arithmetic mean* of the two velocities–their sum divided by two–and the second their *harmonic mean*–twice the reciprocal of the sum of their reciprocals! Most folk are familiar with the arithmetic mean of two (or more) numbers, but not this second type of mean.

Question: *If Albert can eat trough of burgers in 2 hours and Bilbert a trough of burgers in 3 hours, what is the fastest time the two lads together can devour two troughs of burgers?*

Greek scholars of ancient times were familiar with both the arithmetic and harmonic means of two numbers. They also considered a third: the *geometric mean*. It comes from answering the question:

What is the side-length of a square with the same area as an a-by-b rectangle?

The square with side $\sqrt{ab}$ does the trick.

To summarise, we have:

For two numbers a and b:

Arithmetic mean: $A(a,b)=\frac{a+b}{2}$

Harmonic mean: $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$

Geometric mean: $G(a,b)=\sqrt{ab}$

Each of these means have the property that if a and b are quantities in units u (length, or time, or grams, or furlongs per nanosecond), then so is their mean.

CHALLENGE:
i) Prove that each of these means "scale" properly: That is if we double, or halve, or triple, or multiply by k each of a and b, then their means double, or halve, or triple or multiply by k as well.

ii) Prove that if a is smaller than b, then each of means lies strictly between a and b.

iii) Prove that each of the means listed is commutative: "The mean of a and b is the same as the mean of b and a."

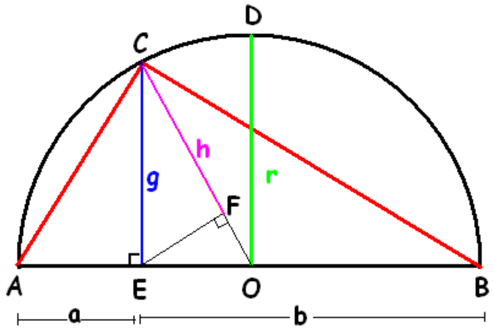
A TOUGH CHALLENGE: Come up with other formulas that satisfy properties i), ii) and iii). For example, the *quadratic mean* $Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}$ works, as does the *contraharmonic mean* $C(a,b)=\frac{a^2+b^2}{a+b}$. How many different examples of means can you create?

SIDE COMMENT: So much of the standard curriculum is obsessed with jargon. We have students learn terms such as *commutative* and *associative property* and *reflexive property* and *transitive property.* Can you name something that isn't reflexive or isn't transitive? Can you name an operation that isn't commutative? How about one that isn't associative? Here's a tricky challenge: *Give an example of an operation on two numbers a and b that is commutative but NOT associative?*

HINT: This newsletter!

I find it curious that we feel compelled to teach students terms and words for concepts we can't easily show could fail to hold. It is only when we are aware of things that don't behave as we expect that we need words for the standard behaviors we do expect!

THE GEOMETRY OF MEANS:
Here is a complicated picture. The green line, the blue line and the purple line in it are of interest to us.



The picture is a circle of diameter $a+b$ with center O. The point E splits the diameter into sections of length a and b and the point C lies directly above E.

The green line OD is a radius and so has length $\frac{a+b}{2}$, the arithmetic mean of a and b.

If we call the length of the blue segment g we have $AC=\sqrt{a^2+g^2}$ and $BC=\sqrt{b^2+g^2}$. If we think of AC as the base, the area of red ΔACB is $\frac{1}{2}\cdot AC\cdot BC=\frac{1}{2}\sqrt{(a^2+g^2)(b^2+g^2)}$. It also equals the areas of ΔAEC and ΔBEC added together. So $\frac{1}{2}\sqrt{(a^2+g^2)(b^2+g^2)}=\frac{1}{2}ag+\frac{1}{2}bg$. Algebra gives $g=\sqrt{ab}$, the geometric mean of a and b.

Call the purple length CF indicated h. Since OC and OA are radii we have $OF=\frac{a+b}{2}-h$ and $OE=\frac{a+b}{2}-a$. By Pythagoras's theorem, $EF^2=g^2-h^2$ and $EF^2=OE^2-OF^2$. Using $g=\sqrt{ab}$

and $g^2 - h^2 = OE^2 - OF^2$, algebra eventually gives $h = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}$, the harmonic mean of a and b.

Thus in the picture of a circle we see the three standard means:

$$r = A(a,b) = \frac{a+b}{2} \quad (green)$$

$$g = G(a,b) = \sqrt{ab} \quad (blue)$$

$$h = H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} \quad (purple)$$

It is clear from the picture that r is longer than g and that g is longer than h. This proves:

If $a < b$ then:

$$a < H(a,b) < G(a,b) < A(a,b) < b.$$

CHALLENGE: Prove that the length ED in the picture is $Q(a,b)$, the quadratic mean of a and b. Where does $Q(a,b)$ fit in the string of inequalities?

ANOTHER CHALLENGE: We have focused on the means of just two numbers a and b. What is the arithmetic mean of three or more values? The geometric mean? The harmonic mean? Once you have the right definition for each of these means for more than two terms, prove that the same string of inequalities holds between them!

TOUGH CHALLENGE: In this sequence of fractions each term is the (arithmetic) average of the two before it:

$$0 \quad 1 \quad \frac{1}{2} \quad \frac{3}{4} \quad \frac{5}{8} \quad \frac{11}{16} \quad \frac{21}{32} \quad \ldots$$

Are the numbers in this sequence approaching a specific value?

RESEARCH CORNER:
Average Ages

To answer the opening "average age" puzzle … Jake, as with every student in that circle, is the same age as Betty. Here's why:

Betty can't be sitting between two students who are each younger than 9 (otherwise "9" would not be the average of those two ages) nor between two neighbours each older than 9. Thus one of Betty's neighbours has age $9-x$ for some value x and the other $9+x$.

But "$9+x$" is the average of 9 and one other age, which must be $9+2x$. And "$9+2x$" must be the average of $9+x$ and one other age, which must be $9+3x$. And so on.

Going around the circle starting with Betty we see that the students' ages are $9, 9+x, 9+2x, \ldots, 9+39x$ with $9+40x$ bringing us back to Betty's age. Consequently $9+40x=9$, giving $x=0$, showing that all students are 9 years old.

Here is the research to be done:

i) What if each student in that circle had age equal to the average age of the two people two her left? Must all students again be the same age?
ii) What if the age of each student equaled the average of the age of the person immediately to her left and the person two places to her right?
iii) What "patterns of average" force all students in a circle to have the same age? How much is this dependent on the size of the circle?

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