WHIZBANG COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2013

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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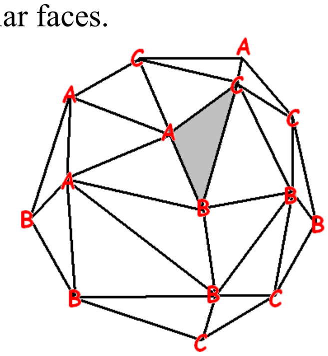
Check out Sol Lederman's podcast page: http://wildaboutmath.com/category/podcast/ Listen to spectacular interviews with mathematical greats such as Keith Devlin, Ian Stewart, Steven Strogatz, and more. Hear their stories and joys in doing and thinking mathematics. Sol's blog makes mathematics accessible, real and personal.

PUZZLER: I have three-dimensional ball-like object composed of many flat triangular faces.



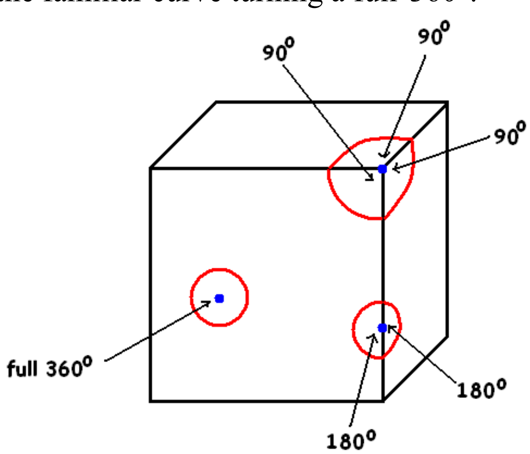
With a marker I randomly labeled each of its corners either A, B or C. Glancing at what I have done, I notice that one of the faces is "fully labeled" - each of the letters A, B and C appears in the corners of that face. Explain to me why, if I look further, I am sure to find a second fully-labeled ABC face.

HOW ROUND IS A CUBE?

One can draw a circle on any surface by pinning one end of a rope into the surface (at the desired center for the circle), pulling the rope taut and swinging that taut rope all the way around, tracing the path of the second end of the rope with a marker.

The circles we draw on the surface of the Earth with rope and sidewalk chalk are relatively small and look just like the ideal images we have in our minds of "flat" circles drawn on perfectly flat surfaces. But if the Earth were the shape of the cube, we could not fail to notice different characters to some of the circles we draw.

For a small circle with center on one of the faces of the cube, the circle we see is the familiar curve turning a full 360°.



With center on the edge of a cube, the circle we see comes in two sections, with each segment turning through an angle of 180°. The total "turning" here is still 360°. (It appears as $180^\circ + 180^\circ$.)

But for a small circle with center a corner of the cube we have three segments each turning through 90°, adding to a total turn of only 270°. This is $\frac{1}{4}$ of a turn short of a full turn.

These deficits represent some kind of measure of the "pointiness." Let's say:

Each corner of the cube has pointiness $\frac{1}{4}$ *(one quarter of a full turn that is).*

As there are eight corners we'll go further and say:

The total pointiness of a cube is $8 \times \frac{1}{4} = 2$ *full turns.*

EXPLORATION: Pi is defined as the circumference of a circle divided by its diameter: $\pi = \frac{C}{D} = \frac{C}{2r}$. We like to believe this ratio is constant for circles on flat surfaces, with value about 3.141. (See the January 2013 Curriculum Essay for commentary on this!)

Let's explore the value of pi for circles on the surface of a cube with center one of the corners of the cube. For ease, assume the side-length of the cube is 1.

Case $0 < r < 1$:
A corner circle with radius less than one comes in three sections, looking like the corner circle we've drawn to the left. Its perimeter is three-quarters the length we expect. Consequently its value of pi is:

$$\frac{\frac{3}{4}C}{2r} = \frac{3}{4} \cdot \frac{C}{2r} = \frac{3}{4}\pi \approx 2.356.$$

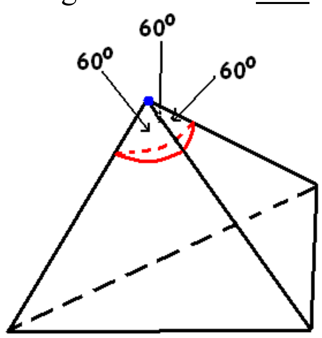
Case $1 \le r < \sqrt{2}$:
What does a corner circle with this radius look like? (Why the number $\sqrt{2}$?) What is the ratio "perimeter to $2r$" for these circles? Does it change as r changes?

Case $r \geq \sqrt{2}$ **:**
How large can r be? What do these larger circles look like? What can you say about their pi-values?

Other circles? How does the value of pi change as r grows for circles with center on the face of the cube? On an edge?

ASIDE: OTHER POINTY SHAPES

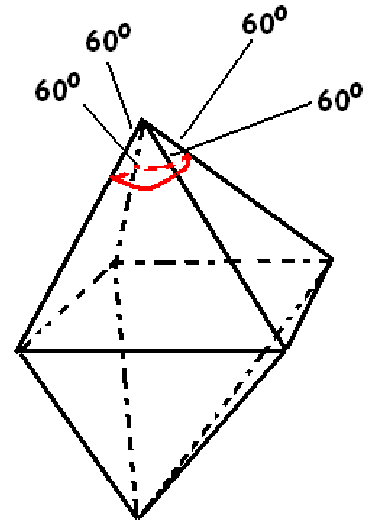
A **regular tetrahedron** has four-equilateral triangular faces with three faces meeting at each of its <u>four</u> corners.



Thus three 60° angles meet at each corner showing that the total amount of turning about any corner is just $3 \times 60^\circ = 180^\circ$. This is $\frac{1}{2}$ a turn short of a full turn.

The total pointiness of a regular tetrahedron is $4 \times \frac{1}{2} = 2$ *full turns.*

A **regular octahedron** has eight-equilateral triangular faces with four faces meeting at each of its <u>six</u> corners.



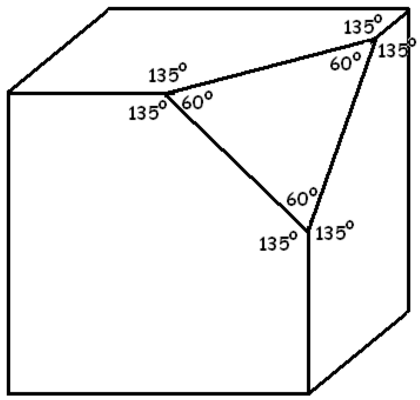
Four 60° angles meet at each corner showing that the total amount of turning about any corner is just $4 \times 60^\circ = 240^\circ$, which is $\frac{1}{3}$ a turn short of a full turn.

The total pointiness of a regular octrahedron is $6 \times \frac{1}{3} = 2$ *full turns.*

Question: What is the total pointiness of each of the two remaining Platonic solids: an icosahedron with 20 equilateral triangular faces, five meeting at each corner, and a dodecahedron with 12 regular pentagonal faces, three meeting at each corner?

ROUNDING THE CUBE

Let's make the cube less pointy by shaving down one of its corners to make a new equilateral triangular face.



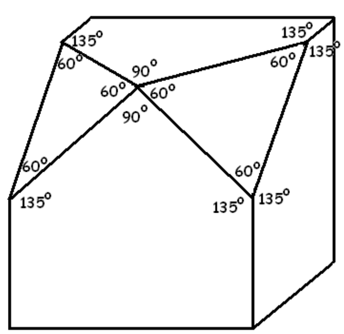
The angles about each new corner sum to $135+135+60=330^\circ$, and so a circle centered on such a corner is deficient by 30°, that is, $\frac{1}{12}$ of a turn. There are three of these corners, plus the seven corners we've left untouched. The total pointiness of the shaved cube is:

$$7\times\frac{1}{4}+3\times\frac{1}{12}=2.$$

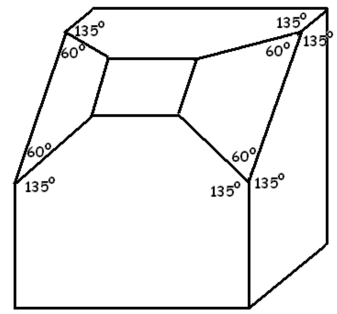
Hmm. Just as pointy!

Shaving off another corner we find the total pointiness is …

$$6\times\frac{1}{4}+4\times\frac{1}{12}+1\times\frac{1}{6}=2.$$ Again!



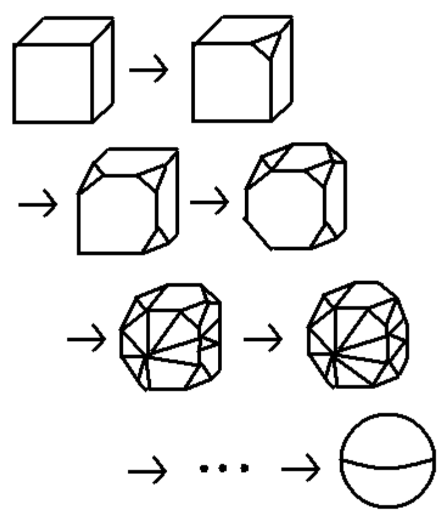
Challenge: Shave off the front corner and make a new square face.



Show that the total pointiness of this figure is still 2.

If we keep shaving off the corners of a three dimensional solid to make it less pointy, do we fail? Is the total pointiness of a figure unchanged by our attempts to make it rounder?

This is a perturbing idea! If true, it means we can keep shaving a cube to make it more and more sphere-like. If the total pointiness never changes, then we can only conclude that the total pointiness of a sphere is also 2 and, moreover, that, in some sense, **a cube is just as round as a sphere!**



TOTAL POINTINESS ALWAYS 2?

This would be mighty weird!

A BETTER WAY TO COUNT TOTAL POINTINESS

Our definition of "pointiness" at a corner is somewhat contorted: We add up the angles around the corner and compute how far off that total is from a full turn of 360°. The pointiness is then that deficit expressed as a faction of 360°.

$$\textit{Angles around a corner} = 360^\circ - x360^\circ$$
where x is the pointiness of that corner.

[For example, at the corner of a cube the pointiness is $1/4$ and the angles sum to:
$270^\circ = 360^\circ - \frac{1}{4}360^\circ$.]

The "total pointiness" of any figure is the sum of all these x-values over all the corners of the figure.

Suppose there are V corners in all (mathematicians prefer to call corners *vertices*). If corner 1 has pointiness x_1, corner 2 has pointiness x_2, and so on all the way up corner V has pointiness x_V, then the total pointiness is given by the sum $x_1 + x_2 + \cdots x_V$. And if we sum all the formulas:

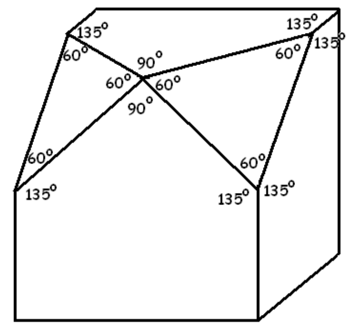
$$\begin{aligned} &\textit{Angles at corner 1} = 360^\circ - x_1 360^\circ \\ + \; &\textit{Angles at corner 2} = 360^\circ - x_2 360^\circ \\ &\vdots \\ + \; &\textit{Angles at corner V} = 360^\circ - x_V 360^\circ \end{aligned}$$

we get:

Sum of all angles over all corners
$=$
the number 360 *summed* V *times*
$-\left(x_1 + x_2 + \cdots + x_V\right)360$

We can use this to get a formula for $x_1 + x_2 + \cdots x_V$, the total pointiness.

Summing all angles over all corners is just another way of saying: *sum all the angles you see in the polyhedron.* For example, in this figure:



there are:

Two triangular faces each with three angles of sixty degrees. (In fact, any triangular face will give three angles adding to 180°.)

Two quadrilateral faces each with four angles of ninety degrees. (In fact, any quadrilateral face will give four angles adding to 360°.)

Four pentagonal faces each with five angles adding to 540°. (In fact, any pentagonal face will give five angles adding to 540°.)

So the sum of all the angles we see is:
$2 \times 180^\circ + 2 \times 360^\circ + 4 \times 540^\circ = 9 \times 360^\circ$.

There are $V = 11$ corners, so according to our formula this number equals:

$$360^\circ \times 11 - (x_1 + x_2 + \cdots + x_{11})360^\circ$$

We see again that the total pointiness is $x_1 + x_2 + \cdots x_{11} = 2$.

GENERALISING …

If f_3 is the number of triangular faces (each with three angles adding to 180°), f_4 the number of quadrilateral faces (each with four angles adding to 360°), f_5 the number of pentagonal faces (with angles adding to 540°), f_6 the number of hexagonal faces (with angles adding to 720°), and so on, then the sum of all angles we see is:

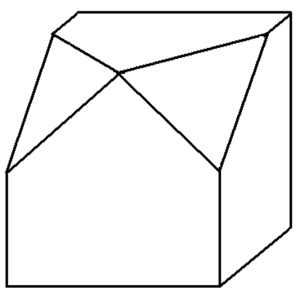
$$180f_3 + 360f_4 + 540f_5 + 720f_6 + \cdots$$

and this is meant to equal:
$360V - (x_1 + x_2 + \cdots + x_v)360$.

Solving for $x_1 + x_2 + \cdots x_V$ gives:

Total Pointiness
$$= V - \frac{1}{2}\left(f_3 + 2f_3 + 3f_4 + 5f_6 + \cdots\right)$$

This formula shows that to count the total pointiness of a polyhedron we need not worry one whit about angles! Just count the number of corners and the number of faces of each type.



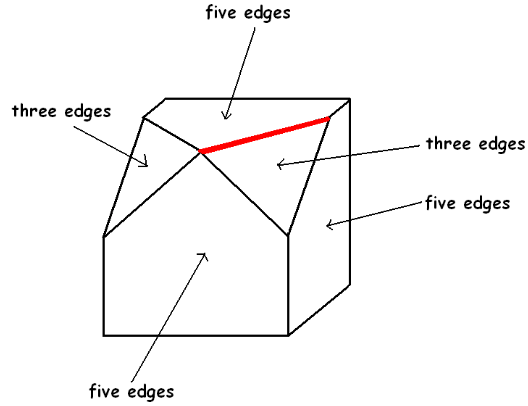
$V = 11$
$f_3 = 2,\ f_4 = 2,\ f_5 = 4,\ f_6 = 0,\ f_7 = 0, \ldots$

But we can do better!

Let F be the total number of faces. (so $F = f_3 + f_4 + f_5 + \cdots$, the number of triangular faces plus the number of quadrilateral faces plus the number of pentagonal faces and so on) and let E be the number of edges we see on the polyhedron.

Each triangular face has 3 edges, each quadrilateral face has 4, each pentagonal face as 5 edges, so it seems that the total number of edges is:

$$3f_3 + 4f_4 + 5f_5 + \cdots .$$



But each edge here is counted twice. For example, the highlighted red edge is counted once as part of the five edges of the top pentagon and once again as one of the three edges of the triangular face. So actually, we have:

$$3f_3 + 4f_4 + 5f_5 + \cdots = 2E$$

Subtracting $f_3 + f_4 + f_5 + \cdots = F$ from this formula gives:

$$2f_3 + 3f_4 + 4f_5 + \cdots = 2E - F$$

and one more time gives:

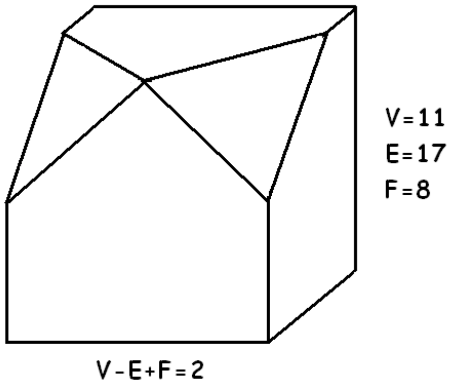
$$f_3 + 2f_4 + 3f_5 + \cdots = 2E - 2F$$

and this is the expression that appears in our equation for total pointiness. We get:

$$V - \frac{1}{2}\left(f_3 + 2f_3 + 3f_4 + 5f_6 + \cdots\right)$$
$$= V - \frac{1}{2}\left(2E - 2F\right) \quad = \quad V - E + F$$

Total Pointiness
$= V - E + F$

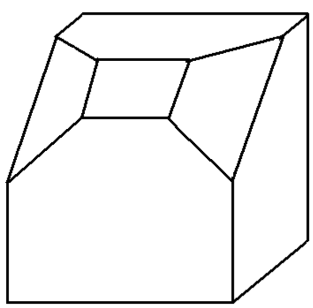
This is spectacular! To compute the total pointiness of a polyhedron, the sum of the pointiness of each of the corners, worry not one whit about angles and worry not one whit about keeping track of the shapes of the faces. Just count the number of corners, the number of edges, and the total number of faces of any type. Then plug these counts into the formula $V - E + F$. Voila! You're done!



The pointiness of the corners add to two full turns

SHAVING CORNERS DOES NOT HELP!

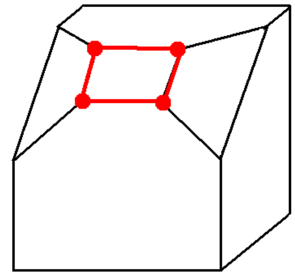
For the figure shown above we have $V - E + F = 2$. This the sum of the pointiness of each of the 11 corners. Suppose we shave the front corner and create a new square face.



We have certainly changed the count of corners, edges and faces, and so the value of $V - E + F$, the total pointiness, may have changed.

Let's check.

We've created four new corners and four new edges (shown in red).



But we have also lost a corner (the one we've shaved off), but have added a new face. So each of our numbers have changed as follows:

$$V \to V + 4 - 1$$
$$E \to E + 4$$
$$F \to F = 1$$

Consequently the value of the formula $V - E + F$ changes to

$$(V + 4 - 1) - (E - 4) + (F - 1)$$
$$= V - E + F$$

That is, it is unchanged!

In fact, one can check that whenever one shaves off a corner, if the face created has k new edges and k new corners, then

$$V \to V + k - 1$$
$$E \to E + k$$
$$F \to F = 1$$

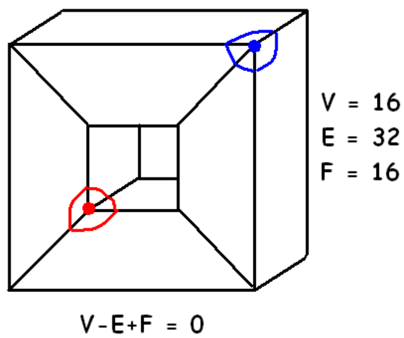
and so the value of $V - E + F$ is unaltered.

It is true … **The total pointiness of a polyhedron does not change by shaving corners.**

In this sense, **a cube is indeed just as round as a sphere!**

NOT ALL SHAPES ARE SPHERE-LIKE!

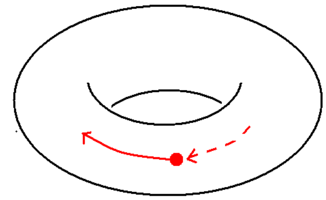
It takes some imagination to come up with an example of a polyhedron for which $V-E+F$, its total pointiness, is not 2. I apologise for spoiling your fun of trying to think of one on your own by showing you this picture now.



This example has zero total pointiness, which is curious. The corner with the blue circle certainly has an angle deficit, and so has positive pointiness. For the total sum of pointiness to be zero, there must be some corners with an excess of angles and so have negative pointiness. The red circle, for example, has more than 360° of turning.

By the way … Since the value of $V-E+F$ is zero and shaving corners does not change this value, we have proved that a polyhedron with a hole is fundamentally different from a polyhedron without a hole. No shaving or the reverse of shaving (what would that be exactly?) will ever turn a donut-shaped polyhedron into a sphere-shaped polyhedron.

CHRISTOPHER COLUMBUS:
By sailing west to return to his start from the east, Christopher Columbus hoped to prove that the Earth was round. The trouble is, such a feat does not prove the Earth is actually a sphere: Columbus would still return to start if the Earth were the shape of a donut, for example!

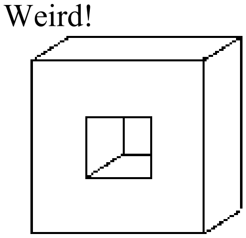


Suppose Columbus decided to cover the Earth with triangles and count the number of corners V, edges E and faces F. What might he conclude if $V-E+F$ turned out to be zero?

HARD STUFF!

We haven't actually proved enough in this essay to really help out Christopher Columbus. Is it true that if $V-E+F=0$ then the polyhedron at hand has a single hole in it like a donut? (Could it be a double-donut? Or some other shape we are yet to dream of?) Are all polyhedrons with $V-E+F=2$ equivalent to spheres?

Does it matter what shape the faces are? For example, in the figure below $V=16$, $E=24$, $F=10$ and $V-E+F=2$, even though this polyhedron really is no different in structure from what we first presented on this page. Weird!



Should faces themselves with holes in them be disallowed?

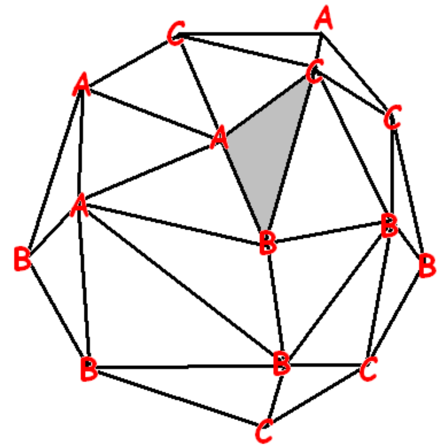
If two polyhedra (with nice faces, whatever that means) have the same value $V-E+F$, is it possible to convert one into the other by a sequence of steps of shaving corners and the reverse of shaving corners (whatever that is)?

These are classical questions that have been explored by some mathematical greats. The quantity $V-E+F$ is called the *Euler Characteristic* of a polyhedron, in honor of Swiss mathematician Leonhard Euler (1707-1783). The "total pointiness" of a three-dimensional shape is called the *total curvature* of the surface, as coined by Carl Friedrich Gauss (1777-1855). We've discovered in this essay that the total curvature of a sphere is two full turns.

Mathematicians prefer to think of one full turn as "2π radians", and so the total curvature of a sphere is 4π. As a sphere of radius R has surface area $4\pi R^2$ and each point on the surface of a sphere seems just as "pointy" as any other, it seems appropriate to say that the pointiness at any one spot on the surface of the sphere is the average value $\frac{4\pi}{4\pi R^2}=\frac{1}{R^2}$. This value is called the *Gaussian curvature* of a point on the sphere of radius R. Gauss managed to figure out meaningful ways to compute the curvature at points on all types of smooth surfaces and link that curvature to angles in triangles on those surfaces.

The story of surfaces and their structure is vast and rich. Dare I say it? ...We have only skimmed the surface of this story.

THE OPENING PUZZLER:



Imagine each triangular face to be a room. Each edge with the label A at one end and B on the other is a door to walk though. All other edges are walls.

Start in your ABC room and walk through its AB door. Walk as far as you can. What do you notice?

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