

★ WOOO! COOL MATH! ★

CURIOUS MATHEMATICS FOR FUN AND JOY



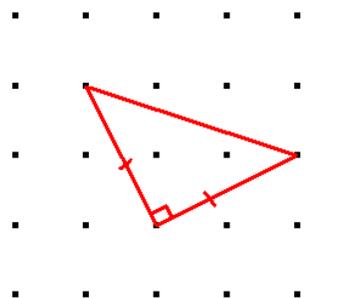
DECEMBER 2013

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

Have you seen Gordon Hamilton's Math Pickle website: www.mathpickle.com ? Top notch and innovative content for K-12 educators and their students.



Puzzler: It is possible to draw a $45-90-45$ triangle on a square lattice of dots with each corner of the triangle at a lattice point. (Non-tilted examples exist too!)

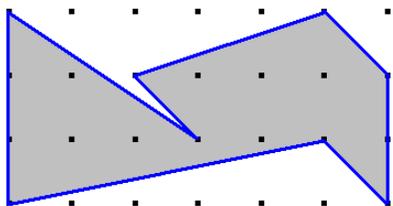


Is it possible to do the same for a $30-60-90$ triangle?

Aside: See this month's Curriculum Essay for my comments on teaching special triangles to students.

LATTICE POLYGONS

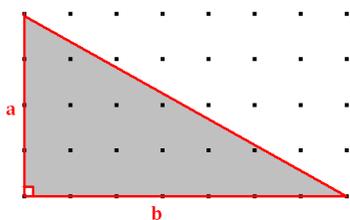
A polygon drawn in a regular square lattice of points with each corner of the polygon landing on one of its lattice points is called a *lattice polygon*. We can imagine the lattice to be the set of points with integer coordinates in a Cartesian coordinate system.



After playing with the opening puzzler one comes to suspect that there is deep and interesting mathematics to be considered on the matter of lattice polygons.

Exploration 1: *Compute the areas of lattice triangles.*

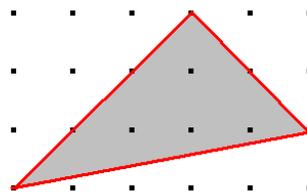
Lattice right triangles with horizontal and vertical legs are easy to analyze.



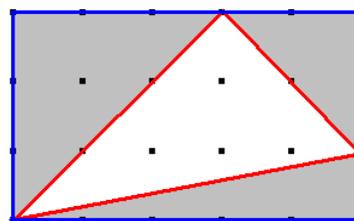
$$\text{area} = \frac{1}{2}ab$$

[**Note:** As the legs of these triangles have integer lengths, the area of any such right lattice triangle is either an integer or a half integer.]

But how does one compute the area of a less cooperative lattice triangle?



It takes an epiphany to think of enclosing any given lattice triangle in a rectangle.



It is easy to compute the area of the rectangle and the areas of the three right triangles that surround the given triangle. Subtraction then gives the area we seek. For example, the area of triangle shown is $3 \times 5 - \frac{1}{2} \cdot 3 \cdot 3 - \frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2} \cdot 5 \cdot 1 = 6$.

We can make a key observation:

The area of any (horizontal and vertical) lattice rectangle is an integer and the areas of the right triangles we consider are integers or half integers. We thus conclude:

The area of every lattice triangle is either an integer or a half integer.

Question: Is it possible to find a tilted lattice rectangle with non-integer area?

As every lattice polygon can be subdivided into lattice triangles (is this obvious?) we conclude:

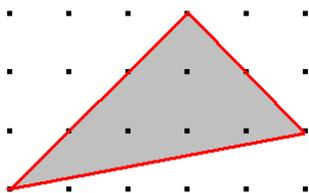
The area of every lattice polygon is either an integer or a half integer.

Question: What is the area of the lattice polygon shown in the first column?

Aside: In 1899 George Pick made and proved an astounding observation about lattice polygons. Your students can too!

Draw a lattice polygon and compute its area. Also count the number of lattice points that lie inside the polygon and the number of lattice points that lie on the boundary of the polygon.

For example, this lattice triangle of area 6 has 4 interior points and 6 boundary points.



Collect data for many polygons.
What do you observe from your data?

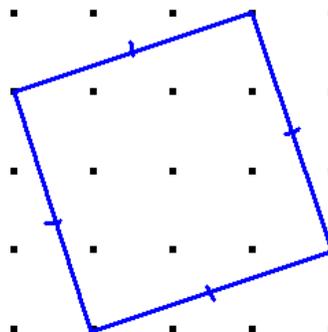
Pick's Theorem is easy to find on the internet if you are curious to see the result.

In 2009 my students discovered Pick's theorem for themselves, managed to prove the result, AND, moreover, extend the result to cover the cases of polygons possessing holes, having "tendrils" and other weird features. Their work appears as *Appendix II* of *Mathematics Galore!* published by the MAA in 2012.



EQUILATERAL LATTICE POLYGONS

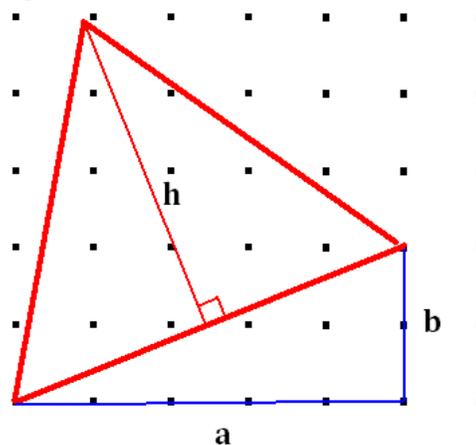
A polygon is called *equilateral* if each of its sides is the same length. It is easy to draw equilateral lattice 4-gons.



Question: The lattice square shown has area 10. It is very easy to draw lattice squares of areas 1, 4, 9, 16, 25, Is it possible to draw a lattice square of area 2? How about one of area 5? Of area 17? Are there any numbers N for which it is not possible to draw a lattice square of area N ? (This final question could be the topic of another Cool Math Essay in, and of, itself!)

Exploration 2: Draw an equilateral lattice triangle.

I am failing to draw one in the picture below. (The apex is not a point with integer coordinates.)



But the diagram shows that the side-length of any equilateral lattice triangle is a number of the form $\sqrt{a^2 + b^2}$ for some integers a and b . (Actually, this is true for the length of any lattice line segment.) Following the work of the Curriculum Math Essay, we have that the height of the equilateral triangle is $\frac{\sqrt{3}}{2} \cdot \sqrt{a^2 + b^2}$ and so its area is:

$$\begin{aligned} & \frac{1}{2} \cdot \sqrt{a^2 + b^2} \cdot \frac{\sqrt{3}}{2} \sqrt{a^2 + b^2} \\ &= \frac{a^2 + b^2}{4} \cdot \sqrt{3} \end{aligned}$$

This is a fraction times $\sqrt{3}$ and is sure to be an irrational number.

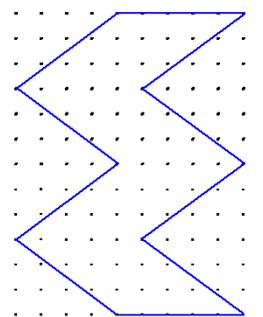
But the area of any lattice triangle is an integer or half integer. Thus we conclude:

It is impossible to draw an equilateral triangle on a square lattice.

Question: So ... Is it possible to draw a 30 – 60 – 90 lattice triangle?

Exploration 3: *Is it possible to draw a five-sided equilateral lattice polygon? A six-sided one? For which values N is it possible to draw an equilateral lattice N -gon?*

Giving half the answers away, the following diagram shows that it is possible to draw an equilateral lattice N -gon for any even number $N > 2$. (This picture makes use of the happy fact that $\sqrt{3^2 + 4^2} = 5$.)



Whether or not equilateral lattice odd-gons exist we'll leave hanging until the end of this essay.

LET'S CHEAT!

Actually it's not cheating at all. It's being realistic and practical!

Have you noticed that every dot we draw in our diagrams actually has some width? Suppose each dot in our square lattice of dots is a disc 0.001 cm wide centered about a point with integer coordinates. Then, YES,...

It is possible to draw an equilateral lattice triangle on a square lattice of dots!

Here's how. First notice that an equilateral triangle of side-length 4262 units has height $2131\sqrt{3} = 3691.000271\dots$ units. Thus if we draw the equilateral triangle with vertices at coordinates $(-2131, 0)$, $(2131, 0)$ and $(0, 2131\sqrt{3})$, all corners lie on a dot of the square lattice of dots!

HARD CHALLENGE: Suppose each of the dot of the square array is only 0.0000001 cm wide. Show that it is still possible to draw an equilateral triangle with each of its vertices on a dot. (Can it always be done no matter how small the width of the dots?)

HARDER CHALLENGE: Show it is possible to draw a regular pentagon on a square lattice of dots (no matter the width of the dots). Show that it is possible to draw an equilateral N -gon on a square grid of dots for any odd number N of your choosing.

DOTS OF ZERO WIDTH

Let's go back to being pure and abstract mathematicians and look at "true" lattice polygons, that is, polygons with corners that have integer coordinates for sure.

We've left the following question hanging: *Is there an equilateral lattice polygon with an odd number of edges?*

We've established that there are no equilateral lattice 3-gons.

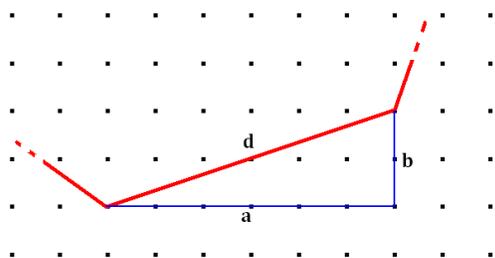
Giving the answer away, let's prove:

There are no equilateral lattice polygons with an odd number of sides.

We'll establish this in steps.

STEP 1: Side-lengths of equilateral lattice polygons in general.

Suppose the side-length of some equilateral polygon is d units.



As we have seen before, d must be a number of the form

$$d = \sqrt{a^2 + b^2}$$

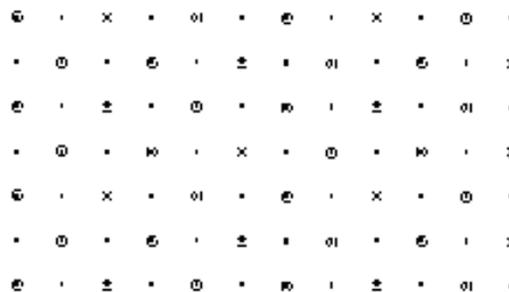
with a and b integers. (One of the integers could be zero.)

Actually, we can write d this way in many different ways: each side of the polygon will likely yield a different pair of integers a and b . For example, with the equilateral polygons with an even numbers of sides on the previous page we used a side-length of $5 = \sqrt{3^2 + 4^2}$ and a side-length of $5 = \sqrt{5^2 + 0^2}$.

If the distance d is an odd number, then d^2 is also odd, which means each of our expressions for $d^2 = a^2 + b^2$ must be of the form $d^2 = (\text{odd})^2 + (\text{even})^2$. If d is even, we have $d^2 = (\text{even})^2 + (\text{even})^2$ or $d^2 = (\text{odd})^2 + (\text{odd})^2$ each time.

STEP 2: Equilateral lattice polygons with an odd number of sides must have d even.

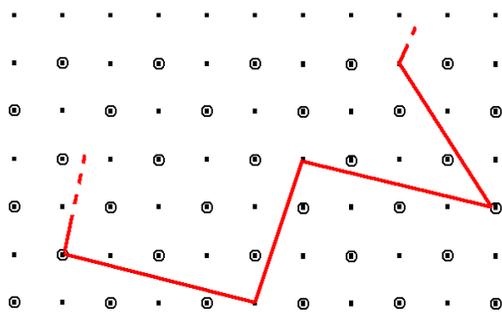
Circle all the dots on the square lattice whose coordinates are both even or both odd. (The origin, for example, is circled. The dot at position $(17, -14)$ is not.)



Now draw an equilateral lattice polygon with an odd number of sides on this lattice.

Because the polygon has an odd number of sides it is impossible for the vertices of the polygons to alternate circled/un-circled all the way around the polygon.

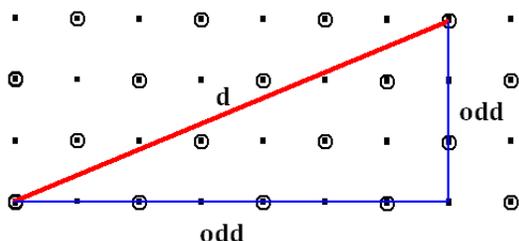
(Only an even number of sides will allow for such an alternating pattern.)



Thus there are two neighboring vertices of the polygon that are either both circled or both uncircled.

For such a side we see that d is of the form $d^2 = (odd)^2 + (odd)^2$ or

$$d^2 = (even)^2 + (even)^2.$$

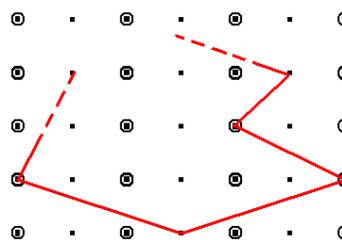


(This picture illustrates the case with both endpoints circled, with odd vertical and horizontal lengths. Pairs of even lengths occur too, and the same applies to case with endpoints uncircled.)

Either way, d is even.

STEP 3: Equilateral lattice polygons with an odd number of sides must have $d^2 = (even)^2 + (even)^2$. We shall never see $d^2 = (odd)^2 + (odd)^2$.

This time circle all the dots in the square lattice with even x -coordinates.



Again, because the polygon has an odd number of sides, there must be a pair of neighboring vertices with x -coordinates both even or both odd. Either way their difference is even, which shows that d is of the form:

$$d^2 = (even)^2 + (something)^2.$$

From step 2, we know that we never have $d^2 = (even)^2 + (odd)^2$, so this means that $d^2 = (even)^2 + (even)^2$ for this one side of the lattice polygon.

That $d^2 = (even)^2 + (even)^2$ at least once shows that d^2 is a multiple of four. (For some integers some integers n and m , $d^2 = (2n)^2 + (2m)^2 = 4(n^2 + m^2)$.)

But notice that any expression of the form $(odd)^2 + (odd)^2$ is not a multiple of four:

$$\begin{aligned} d^2 &= (2p+1)^2 + (2q+1)^2 \\ &= 4(p^2 + q^2 + p + q) + 2 \end{aligned}$$

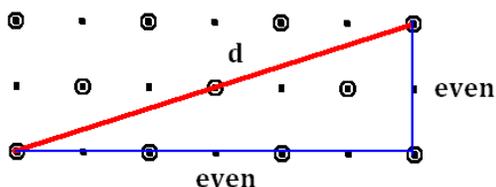
Thus in all the way we shall see d^2 as a sum of two squares, it can only ever be in the form $(even)^2 + (even)^2$.

STEP 4: All corners of the equilateral lattice polygon have even coordinates.

We can always shift the polygon so that one of its corners is at the origin.

This picture shows that to move from one corner of the polygon to the next, we always add an even number to the x -

coordinate of the first point and an even number to its y -coordinate.



If we start this thinking at the origin $(0,0)$ and march around the polygons' boundary recording the coordinates of the vertices, we see that all coordinates are even numbers.

STEP 5: Shrink the equilateral lattice polygon.

Since all the coordinates of our supposed equilateral lattice polygon are even, we can divide them each by two to produce a scaled version of the original equilateral lattice polygon. It is an equilateral lattice polygon in its own right at half the scale.

Now repeat steps 1, 2, 3, and 4 for this smaller polygon to argue that it too must have even coordinates! Divide them each by two to produce another equilateral lattice polygon. By steps 1, 2, 3, and 4 its coordinates are even, and we can divide them each by two yet again. And again, and again, ad infinitum.

If an odd sided equilateral lattice polygon exists we have just argued that there are integers (its coordinates) that can be divided by two infinitely often and remain integers. This is absurd. We can only conclude that odd-sided equilateral lattice polygons do not exist!

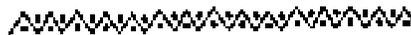
Challenge: It is known that rectangles and octagons are the only possible equiangular lattice polygons. Can you draw an equiangular octagon? Can you prove that no other equiangular objects exist?

Challenge: Imagine an equilateral triangular array of points. (Here points have coordinates of the form $\left(\frac{m}{2}, \frac{n\sqrt{3}}{2}\right)$

with m and n integers that are both even or are both odd.) It is possible to draw an equilateral triangle on such an array. Prove it is not possible to draw a square on it. (But show it is possible if each point is actually a dot of finite width!)

Research: For which N does there exist an equilateral N -gon on a triangular array of points? An equiangular N -gon?

More Research: Investigate the existence of equilateral and equiangular lattice objects in three-dimensional cubic arrays of points. (Is it possible to draw an equilateral triangle tilted in space?) What does a three-dimensional triangular array of points look like?



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