# WE GOT COOL MATH!

## CURIOUS MATHEMATICS FOR FUN AND JOY

**DECEMBER 2012**

**PROMOTIONAL CORNER:** *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

***

Interested in teaching problem solving in our mathematics classroom in such a way that doesn't distract from the demands of the curriculum and directly attends to the Common Core State Standards? Have a sneak peek at materials being developed along these lines at www.jamestanton.com/?p=1193. The MAA and I will be developing professional development opportunities plus books, videos, and more. Big plans afoot!

Looking for a geometry text, one that cuts through the clutter and works to promote joyous mathematical thinking? Three high-schools are currently using

***GEOMETRY Vol I and Vol II***

available at www.lulu.com (search under "Tanton Geometry").

***

And while you are at lulu.com, have a look at:

***THINKING MATHEMATICS: Vol 1***
***Arithmetic = Gateway to All***

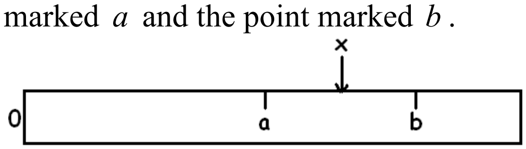
I am particularly chuffed with this first volume in the series. This is written for a beginning audience (high-school age and up) and really promotes deep understanding, thinking, reflection, exploration, and wonder; finding incredible depth in the simplest of ideas.

**PUZZLER:** Darian is about to write sixty-four numbers in the cells of an $8\times 8$ chessboard, one number per cell, in such a way that each number equals the arithmetic average of the numbers in its horizontal and vertical neighbours. (Corner cells have two neighbours, edge cells three neighbours, and interior cells four neighbours.) For example, if Darian writes $7$ in the top left corner cell, he could then write $4$ and $10$, say, in each of its two neighbours. The edge cell with the number $4$ could then have neighbours $7$ (necessarily), $8$ and $-3$, and so on.

As Darian completes this task he notices something very surprising. What?

**THE RETURN OF AVERAGES:**
In the July 2012 Cool Math essay we discussed the appearance of averages in three different settings and learnt that different contexts can lead you to different formulations of what "average" could best mean. Those ideas have stayed in the back of my mind the past number of months. My subconscious was clearly telling me that there is more to say on the matter of "average."

**Geometric Interpretations:**
When asked "what is the average of two numbers $a$ and $b$?" most people think of their arithmetic average: $\dfrac{a+b}{2}$.
On a ruler this represents the location of the point midway between the point marked $a$ and the point marked $b$.

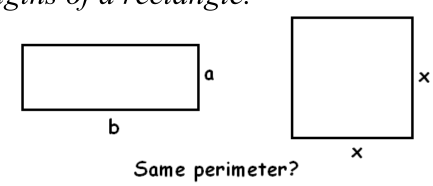



Most people dub this interpretation as "obvious," but actually there are some subtle issues afoot that should be attended to. "$a$" on a ruler is not a number, but a physical length, the actual distance of the point marked $a$ from the left end of the ruler. So $a$ has units of length. Similarly $b$ is a physical length.

The midpoint is a point on the ruler equidistant from the two locations marked $a$ and $b$. If we denote its distance from the left end of the ruler as $x$, then the distance between the points marked $a$ and $x$ is the difference of the lengths, $x-a$. The distance between the points marked $x$ and $b$ is the difference of lengths $b-x$. We want $x-a=b-x$, giving $x=(a+b)/2$, as claimed.

This seems like hyper-pedantic work, but as we shall see attention to units is important when thinking about averages.

Another "length" interpretation: *If the two given numbers $a$ and $b$ are positive, we can think of them as the side lengths of a rectangle.*

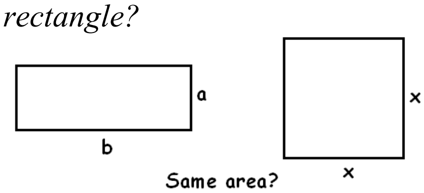



*What is the side-length of a square with the same perimeter as the rectangle?*

Here we need to solve $4x = 2a+2b$ giving $x=\dfrac{a+b}{2}$, the arithmetic mean.

This opens us to another idea:

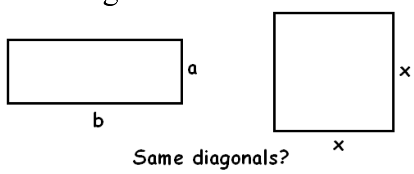
An "area" context: *What is the side-length of a square with the same area as the rectangle?*




Here we need to solve $x^2=ab$ giving $x=\sqrt{ab}$. This formula is a meaningful "average" of the two numbers $a$ and $b$, and is called their geometric mean.

Why stop at this?

Another length context: Suppose I am Pythagophilic and like to focus on diagonal lengths.




*What is the side-length of the square with the same diagonal as the rectangle?*

Here we need to solve

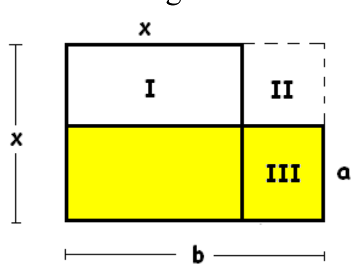
$$\sqrt{a^2+b^2}=\sqrt{x^2+x^2}\ .$$

This gives

$$x=\sqrt{\frac{a^2+b^2}{2}}$$

which is called the quadratic mean of $a$ and $b$. (My thanks to @hubbard_rob for pointing this out to me in a twitter conversation.)

**The fun doesn't stop!** Overlap a rectangle with side lengths $a$ and $b$ and a square of side-length $x$ as shown.




i) Show if $area\ I = area\ III$ then we must have $x=\sqrt{ab}$, the geometric mean of $a$ and $b$.

ii) Show that if $area\ I = area\ II + area\ III$ then we must have $x=\frac{a+b}{2}$, the arithmetic mean of $a$ and $b$.

iii) Show that if $area\ I + area\ II = area\ III$, then we must have $x=\frac{2}{\frac{1}{a}+\frac{1}{b}}$. This is called the harmonic mean of $a$ and $b$. (This arose in the context of studying velocities in the July 2012 essay. Units of "length per time" were important in that context.)

**Research:**
It seems that $area\ II$ is in some dispute as to where it belongs. If we align it with $area\ I$ we get the harmonic mean. If we align it with $area\ III$ we get the arithmetic mean. If we align it with both ($area\ I + \frac{1}{2}area\ II = area\ III + \frac{1}{2}area\ II$) we get the geometric mean. What means result if we split $area\ II$ between the two in some proportion? Say, $area\ I + \frac{1}{3}area\ II = area\ III + \frac{2}{3}area\ II$, or more generally, $area\ I + \lambda \cdot area\ II = area\ III + (1-\lambda)area\ II$, for some value $\lambda$. What does $x$ then equal? What new means do we discover?

**More research!** Apply all these ideas in the third dimension. What cube of side-length $x$ has the same volume as an $a\times b\times c$ rectangular box? Same surface area? Same sum of edge lengths?

Align a cube and a rectangular box in a manner analagous to the diagram on the left. Identify regions of space. Set some equal to others. Can you discover interesting means for the three numbers $a$, $b$ and $c$?

In case you hadn't noticed, the word "mean" is often the preferred word for formula that represents some type of middle value for a set of numbers.

**Comment:** We have lapsed into assuming that our two numbers $a$ and $b$ under consideration are positive. (The geometric mean of two quantities, for instance, could be in trouble otherwise!) Just to be clear, let's say that from now on, for sure, all numbers discussed shall be assumed positive.

## WHAT SHOULD A MEAN DO?

Greek scholars of the time of Pythagoras (ca 500 B.C.E.) identified the four means that have appeared thus far: the arithmetic mean $\frac{a+b}{2}$, the geometric mean $\sqrt{ab}$, the harmonic mean $\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}$, and the quadratic mean $\sqrt{\frac{a^2+b^2}{2}}$. And as the research challenges suggest other means are possible. For example, the <u>contraharmonic mean</u> of two numbers $a$ and $b$ is given by $\frac{a^2+b^2}{a+b}$.

**Challenge:** If $A$, $G$ and $H$ denote the arithmetic, geometric and harmonic means of two numbers, prove that $G=\sqrt{AH}$. Also prove that $H \le G \le A$. (Use areas I, II and III?)

Let $M(a,b)$ denote a formula for some mean of $a$ and $b$.

All the means listed give values that lie between $a$ and $b$. For example, to prove this for the quadratic mean notice that if $a \le b$, then

$$\sqrt{\frac{a^2+b^2}{2}} \le \sqrt{\frac{b^2+b^2}{2}} = \sqrt{b^2} = b$$

and

$$\sqrt{\frac{a^2+b^2}{2}} \ge \sqrt{\frac{a^2+a^2}{2}} = a.$$

If $a \le b$, then $a \le M(a,b) \le b$.

As a consequence, the mean of two identical values is that value: $M(a,a)=a$. (This feels right!)

Also, each of the means listed above is has the same units as the original values. For example, if $a$ and $b$ are lengths, then $a^2+b^2$ is the sum of two areas and so has units of area. Halving this to get $\frac{a^2+b^2}{2}$ is still a result in units of area. Taking the square root, $\sqrt{\frac{a^2+b^2}{2}}$, brings us back to units of length.

The Greeks were very aware of their units (and some historians argue this is why they never invented algebra, per se. An expression such as "$x^2+x$" would have no meaning to the Greeks: How can you add an area and a length?) They felt it necessary that any mean of two numbers, which represents some sort of "middle" of them, should be another value of the same type: the mean of two lengths should be a length, the mean of two weights should be a weight, and so on. Very reasonable!

How does this respect for units translate into mathematics?

The arithmetic mean of $3$ kilograms and $7$ kilograms is $5$ kilograms:

$$M(3,7)=5.$$

If I measured using units of pounds instead, I would need to multiply each value by a factor of 2.2. (There are 2.2 pounds to the kilogram.) The mean should also change by a factor of 2.2 (after all, it too has units of weight).

$$M(2.2\times 3, 2.2\times 7) = 2.2\times 5.$$

In general, we'd expect a mean to satisfy:

$M(ka,kb) = kM(a,b)$ for any value $k$.

Finally, notice that all our means are symmetrical: it does not matter in which order we insert terms into the formula.

$M(a,b) = M(b,a)$.

So we have identified three basic qualities a mean between two values should possess:

If $a \le b$, then:

(1) $a \le M(a,b) \le b$

(2) $M(a,b) = M(b,a)$

(3) $M(ka,kb) = kM(a,b)$

**Comment:** One might desire additional properties. For lengths along a ruler we might feel that a "translational property" should hold: If we shift all our numbers to the right by 2 inches, then the mean should also shift by 2 inches:

$$M(a+2,b+2) = M(a,b)+2\,.$$

Such additional properties will start to restrict our list of possible means. For example, the arithmetic mean satisfies this translational property, but the geometric mean does not.

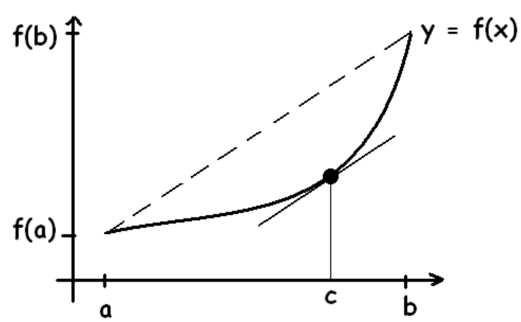
**Challenge:** Show that $\max(a,b)$, the formula that returns whichever of the two numbers is biggest, satisfies the three basic properties of a mean. Show that $\min(a,b)$ does too.

**FINDING LOTS OF MEANS!**
**Means from Calculus.**

In calculus class we learn of the Mean Value Theorem:

*For a differentiable function $f$ on an interval $[a,b]$ there is a value $c$ between $a$ and $b$ at which the instantaneous rate of change of the function equals its average rate of change:*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$




If the function is concave, the value of $c$ is unique.

So for each choice of concave function $f$ we can use the formula $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ to find a value $c$ between any two numbers $a$ and $b$. This number $c$ has something to do with averages so it might just give as new means to consider!

For example, with $f(x) = x^2$ (and $f'(x) = 2x$) the mean-value formula reads:

$$2c = \frac{b^2-a^2}{b-a}$$

giving $c = \dfrac{a+b}{2}$, the arithmetic mean!

Choosing $f(x)=\dfrac{1}{x}$ gives:

$$-\frac{1}{c^2}=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}=\frac{a-b}{(b-a)ab}=-\frac{1}{ab}$$

and so $c=\sqrt{ab}$, the geometric mean.

This mean-value theorem approach yields two familiar means at least.

> **Question:** It is clear that the first property of being a mean will be satisfied most of the time: If $a<b$, then certainly $a\le c\le b$. It is not clear what happens if $a=b$: the formula $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ breaks down.
> Or does it? What happens to the quantity $\dfrac{f(b)-f(a)}{b-a}$ as $b\to a$? What then is the "right" choice for $c$ in the case with $a=b$?
>
> What about the second property of being a mean? Is it automatically satisfied?

We can try all sorts of different (concave) choices for $f(x)$.

Consider $f(x)=\ln x$. Show that this gives $c=\dfrac{b-a}{\ln b-\ln a}$, which is called the logarithmic mean.

**Comment:** The logarithm of a value is said to have no units. It represents the power of a quantity and so is simply a "count" of sorts, counting how many times an arithmetic operation is to be applied. For example, in $10^3$ the 10 might be in units of length, but the "$3$" is referring only to the instruction to multiply this quantity of length by itself three times.

The logarithmic mean satisfies all three properties of being a mean.

Actually this brings up an important point: We have shown that the mean-value theorem will lead to a formula for $c$ that satisfies the first two mean properties. We've skipped over the question of whether or not the third mean property, $M(ka,kb)=kM(a,b)$, automatically holds too.

> **CHALLENGE:** In fact it doesn't! Find a choice for a function $f(x)$ that leads to formula for $c$ for which the third property of being a mean fails.

Carrying on …

Consider $f(x)=x^3$: This gives $c=\sqrt{\dfrac{a^2+ab+b^2}{3}}$. This formula satisfies all three properties.

Consider $f(x)=x^{\frac{1}{2}}$: This give $c=\left(\dfrac{\sqrt{a}+\sqrt{b}}{2}\right)^2$. This satisfies all three properties too.

> **CHALLENGE:** Show that $f(x)=x^r$ for $r\ne 0,1$ is sure to give a formula that satisfies the third property of being a mean. (What happens for $r=0$? For $r=1$?)
>
> Show that as $r\to\infty$, the formula one obtains from $f(x)=x^r$ approaches the formula $\max(a,b)$. (What do you get for $r\to-\infty$?)

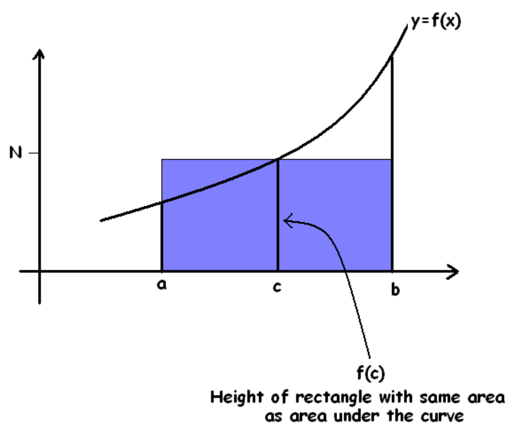
> **CHALLENGE:** Does this approach give all means? Is there a function $f(x)$ that yields the quadratic mean? How about the harmonic mean?

**CURRENT RESEARCH**

Work of this type is still a topic of active research! Just last month Bruce Frank published a paper, "Looking for a few good means," (American Mathematical Monthly, Volume 119, No. 8, Pages 658-669) in which he examined an integral approach to means. This thinking uses the notion of the "average value of a function."

Suppose $y = f(x)$ is a continuous function defined for positive inputs and $a \le b$ are two numbers. Then $\int_a^b f(x)\,dx$ is the area under the curve from $x = a$ to $x = b$.

There is some height $N$ along the $y$-axis so that the rectangle above $x = a$ to $x = b$ along the $x$-axis has the same area as the area under the curve. (Imagine the area being "spread out" evenly to make a level surface at $y = N$.)




Thus $(b-a)N = \int_a^b f(x)\,dx$, giving $N = \frac{1}{b-a}\int_a^b f(x)\,dx$, which is called the average value of the function on $[a,b]$.

If $f$ is an increasing or a decreasing function, the intermediate value theorem assures us there is unique point $c$ along the $x$-axis so that $f(c) = N$. We have:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This gives us a new formula for the hunt of means. For example, using $f(x) = x$ gives $c = \frac{a+b}{2}$, using $f(x) = \frac{1}{x^2}$ gives $c = \sqrt{ab}$, and $f(x) = \frac{1}{x}$ gives the logarithmic mean. (Check these!)

**Question:** Compare the functions used here to the ones we used in the mean-value theorem. Anything curious to note?

**Technical Challenge:** It is clear that if $a < b$, then certainly $a \le c \le b$. But again it is not clear what happens if $a = b$: the formula $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ breaks down.

Use the first fundamental theorem of calculus (and/or L'Hopital's Rule?) to show that

$$\frac{\int_a^{a+h} f(x)\,dx}{(a+h)-a}$$

approaches the value $f(a)$ as $h \to 0$. What then is the "right" choice for $c$ in the case with $a = b$?

What about the second property of being a mean? Does

$$\frac{1}{a-b}\int_b^a f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx?$$

Again the first and second properties of being a mean are sure to automatically hold for any formula we obtain, but the third property is in doubt.

As before, one can show that $f(x)=x^r$ for $r\neq 0$ is sure to give a formula that satisfies the third property of being a mean. The new result in Frank's paper is that the functions $f(x)=x^r$ are the only functions that give formulas satisfying the critical third condition using this integral approach!

**CHALLENGE:** Okay. What's up? What is the relationship between the mean-value theorem for derivatives (the calculus theorem we first mentioned) and the mean value theorem for integrals (the second calculus theorem listed)? They both seem to yield the same results about means. Have we essentially just completed the same work twice, but in two different guises?

**SOLUTION TO THE OPENING PUZZLER:**

The only way for Darian to complete his task is to fill in each and every cell with the same number!

Suppose Darian did start with the number 7. He writes 7s for a while, but as soon as he writes a different number $N$ in a cell neighbouring a 7 he is doomed!

Suppose he writes a larger number. (The case of smaller is analogous.) The new number $N$ he writes has 7 as a neighbor. But $N$ needs to be the average of all its neighbors. Having 7, smaller than $N$, as one of its neighbours means that $N$ requires another neighbor larger than $N$ to "counter act" the $7$. Call this new number $M$. We have $7<N<M$.

Now focus on $M$. It has the smaller number $N$ as a neighbour but needs to be the average of all its neighbours. Thus $M$ has an even larger neighbor still.

In this way we are doomed to follow an ever-increasing path of larger and larger values. As there are only 64 cells, this path must eventually return to a cell already visited, which is a problem, as any revisited cell won't be a value even larger still!

The only way out of this pickle for Darian is to write the number 7 for each and every cell.

**Question:** Can Darian complete the task if he used a mean different from the arithmetic mean? (Oooh …How do you generalize the means we have discussed to more than two values?)

**BONUS PUZZLER:**
I jog uphill at a speed of 4 miles per hour, I jog on flats at 6 miles per hour, and I jog downhill at 12 miles per hour. If it takes me 1 hour to jog to work and 2 hours to jog home from work, how many miles is it to work?

**The real question**… What is special about the numbers "4", "6" and "12" that make this puzzle work?

\


tanton.math@gmail.com