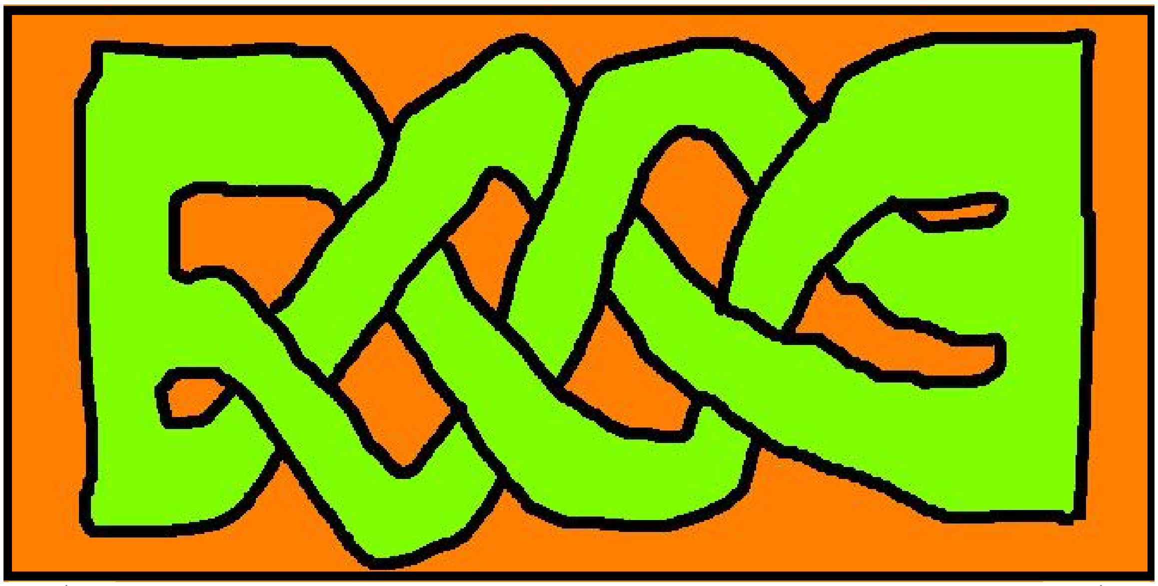

WICKED COOL MATH

CURIOUS MATHEMATICS FOR FUN AND JOY

Welcome to a new Mathematics Letter!

APRIL 2012

Dr. James Tanton is new to the D.C. area, but for the past eight years has been the founding director of the St. Mark's Institute of Mathematics up in the Boston Area.

He is a research mathematician (PhD Princeton, 1994) deeply interested in uniting the mathematics experienced by school students and the creative mathematics practiced and explored by mathematicians. He has worked as a full-time high school teacher and does all that he can to bring joy into mathematics learning and teaching.

James writes math books. He gives math talks and conducts math workshops. He teaches students and he teaches teachers. He publishes articles and papers, always thinking, creating and doing new math. And he shares the mathematical experience with students of all ages, helping them publish research papers too!

Welcome to the first *WICKED COOL MATH LETTER* from the nation's capital!

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A PUZZLER:
Here's a delightful fraction puzzle with a mathematically surprising answer. Try it before reading on. (I give the answer away next in this essay!)

In a particular town two-thirds of the adult men are married to five-eighths of the adult women. What fraction of the entire adult population is married? (Assume each woman is married to one man, and vice versa!)

THE MEDIANT OF FRACTIONS:
Life for young students would be so much easier if the following rule for adding fractions were true:

$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$$

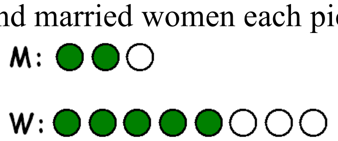
Comment: Actually this rule does work every now and then. For example, $\frac{-1}{3}+\frac{4}{6}$ is indeed $\frac{3}{9}$. Other examples?

Mathematicians call the quantity $\frac{a+c}{b+d}$ the <u>mediant</u> of the two fractions a/b and c/d.

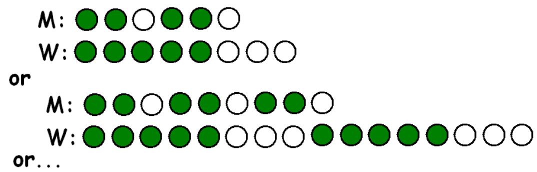
Challenge: Prove that the median of two positive fractions always lies between the two fractions. (What if one of the fractions is negative?)

On occasion the mediant appears as a useful construct in mathematical applications. For example, the marriage problem above relies on computing the mediant of the two fractions mentioned (once you have first converted them to have a common numerator!). To explain:

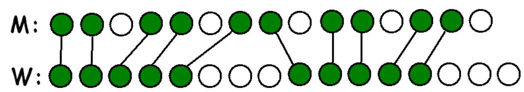
Represent the fraction of married men and married women each pictorially:



Actually any multiple of these diagrams depicts the same proportion of married men and women.

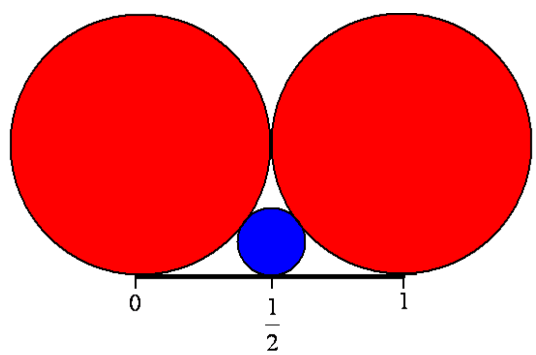


But since each woman is matched with one man, and vice versa, a diagram that shows this matching must have an equal number of green dots on each line. This suggests we should think "$10/15$ of men and $10/16$ of women."



It is now clear, for the entire adult population, 20 out of every 31 people are married. And notice that $\frac{20}{31}$ is the mediant of $\frac{10}{15}$ and $\frac{10}{16}$.

FORD CIRCLES: Two circles of radius one half sitting at the ends of a unit line segment just touch. What is the radius of the circle that fits snuggly below them above the position $x=1/2$ on the segment?



It takes a little work, but one can show that this circle has radius $1/8$. (Try it!)

Now comes a harder question …

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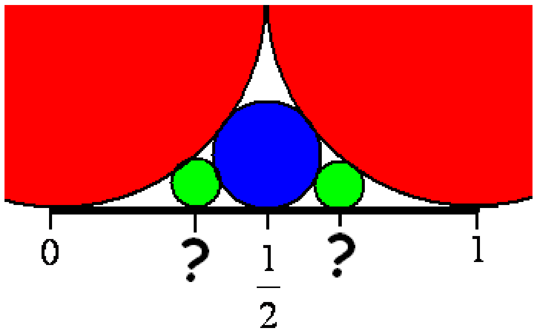
At what positions on the unit segment do the next pair of snugly fitting circles lie? What are the radii of those circles?



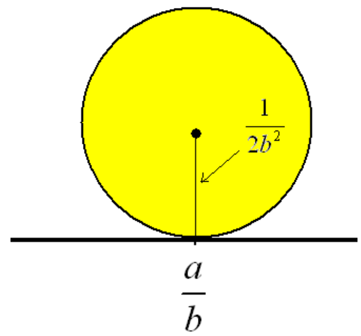
And then where and what size are the next biggest set of circles that fit snuggly above the unit segment? Then same for the next size after that? And after that?

In 1938 American mathematician Lester Ford Sr. explored this question and made the following remarkable discovery.

> *Above the fraction $\frac{a}{b}$ on the unit number segment, written in reduced form, draw a circle of radius $\frac{1}{2b^2}$. Then these circles stack in the way suggested by the problem.*

The first two circles can be regarded as sitting above the fractions $\frac{0}{1}$ and $\frac{1}{1}$ and each indeed have radius $\frac{1}{2\cdot 1^2}=\frac{1}{2}$. The next largest circle sits above $x=\frac{1}{2}$ with radius $\frac{1}{2\cdot 2^2}=\frac{1}{8}$. Given the formula according to Ford's result, the next largest circles have radius $\frac{1}{2\cdot 3^2}=\frac{1}{18}$ occuring at positions $x=\frac{1}{3}$ and $x=\frac{2}{3}$. After this the next largest circles have radius $\frac{1}{2\cdot 4^2}=\frac{1}{32}$ at positions $x=\frac{1}{4}$ and $x=\frac{3}{4}$ (reduced fractions), and then of radius $\frac{1}{50}$ at $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},$ and $\frac{4}{5}$, and so on.

PROVING FORD'S RESULT: If we think of the unit line segment as lying on the x-axis of a coordinate system, then we can see that the circle at position $x=\frac{a}{b}$ has center $C_1=\left(\frac{a}{b},\frac{1}{2b^2}\right)$ and radius $r_1=\frac{1}{2b^2}$.



The circle at $x=\frac{c}{d}$ has center $C_2=\left(\frac{c}{d},\frac{1}{2d^2}\right)$ and radius $r_2=\frac{1}{2d^2}$.

Two circles are tangent precisely when the distance between the centers of the two circles, $|C_1C_2|$, equals the sum of their two radii, r_1+r_2. To avoid square roots, let's square both of these quantities.

$$|C_1C_2|^2=\left(\frac{a}{b}-\frac{c}{d}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2$$

$$\left(r_1+r_2\right)^2=\left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2$$

Expanding each of these and comparing, one can see that these are equal precisely when $\left(ac-bd\right)^2=1$. (This a bit of tedious algebra work!)

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We have: *The Ford circles at* $\frac{a}{b}$ *and* $\frac{c}{d}$ *are tangent precisely when* $ac - bd = \pm 1$.

In fact, if you are truly patient you can see that the algebra says:

$$|C_1C_2|^2 = (r_1 + r_2)^2 + more$$

where $more = \frac{(ac-bd)^2 - 1}{b^2 d^2}$. Since $(ac-bd)^2$ is sure to be a positive whole number (why can't it be zero?) we have:

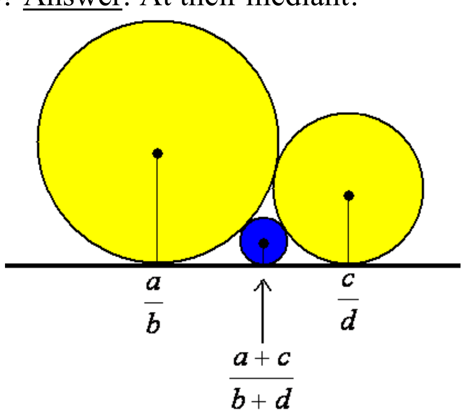
$$|C_1C_2| \geq r_1 + r_2$$

This shows: *No two Ford circles intersect. If they touch, they only touch tangentially.*

This is astounding: **Above each and every one of the infinitely many fractions on the unit number line it is possible to draw a circle so that no two circles ever intersect, only touch.**

RETURN OF THE MEDIANT:

Suppose the circles at $x = \frac{a}{b}$ and $x = \frac{c}{d}$ are tangent. This means $ad - bc = \pm 1$. Where does the circle fitting below them lie? Answer: At their mediant!



To prove this we just need to check that the circle at $\frac{a+c}{b+d}$ is tangent to the one at $\frac{a}{b}$ and to the one at $\frac{c}{d}$. And to do this we need to show that:

$a(b+d) - b(a+c)$ equals ± 1

and

$d(a+c) - c(b+d)$ equals ± 1

Both of these follow from algebra, noting that $ad - bc = \pm 1$.

HARD CHALLENGE: *Prove that every fraction* $\frac{p}{q}$ *between zero and one is the mediant of some pair of fractions* $\frac{a}{b}$ *and* $\frac{c}{d}$*, each with a smaller denominator, satisfying* $ad - bc = \pm 1$.

(WARNING: This is quite tricky!)

This final challenge brings matters full circle(!) If we start with two circles each of radius one half at positions $x = 0$ and $x = 1$ and fill in snug fitting circles below them and above the number line, we produce fractions given by mediants. The final challenge then proves that every fraction shall eventually appear via these mediants to produce the full diagram described by Lester Ford!

INTERNET RESEARCH: Learn about *Farey Sequences* from the internet.

PERSONAL RESEARCH: Six congruent circles sit perfectly about a central circle of the same size. If we start filling in the gaps with snug fitting circles, at what locations on the central circle do they appear?

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