CURRICULUM INSPIRATIONS: www.maa.org/ci

Innovative Online Courses: www.gdaymath.com
Tanton Tidbits: www.jamestanton.com

WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

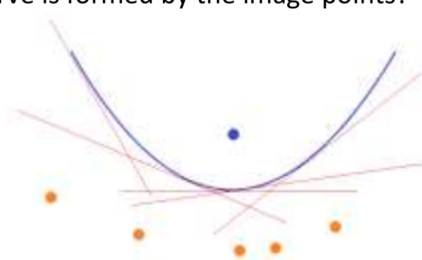
SEPTEMBER 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

***

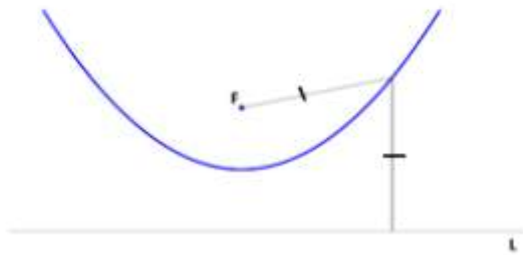
THE ASTOUNDING POWER OF AREA! Check out my educator notes at www.gdaymath.com. (Look for the course by this name.)

OPENING PUZZLE: Suppose we reflect the focus point F of a given parabola across each of its tangent lines. What curve is formed by the image points?



FOLDING A PARABOLA

Given a point F and a line L in the plane, the *parabola* with *focus* F and *directrix* L is the locus of all points P whose distance to F matches its vertical distance to L.



A popular way to construct a parabola is to take a sheet of paper, mark a special point on it, and then fold the bottom edge of the paper up to that point to make a crease. If one does this a large number of times, the outline of a parabolic curve emerges.




Of course, the question arises: How do we know that these fold lines envelop a true parabola? Is it obvious that each crease is a tangent line to this conic curve?

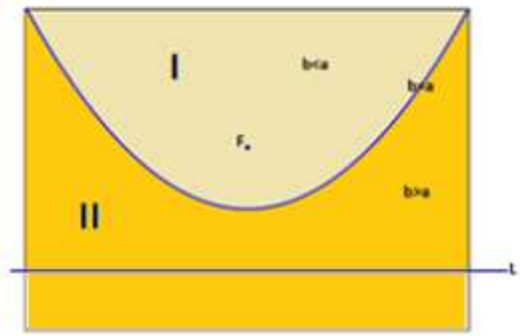
Regions of a parabola:

For each point P in the plane set:

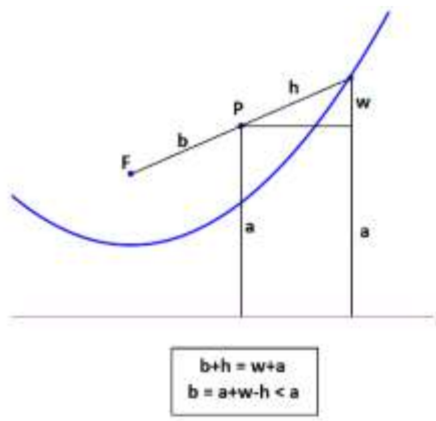
$b = PF$, its distance to F,

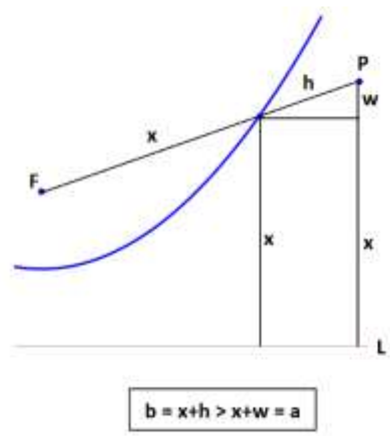
$a =$ the vertical distance from P to L.

The set of all points with $b = a$ define the parabola. The points on the same side of the parabola as the focus (region I) satisfy $b < a$ and the points in the remaining region (region II) satisfy $b > a$.



The following diagrams establish these claims.

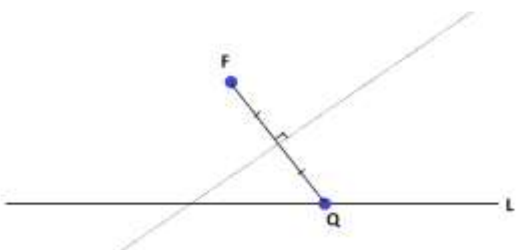




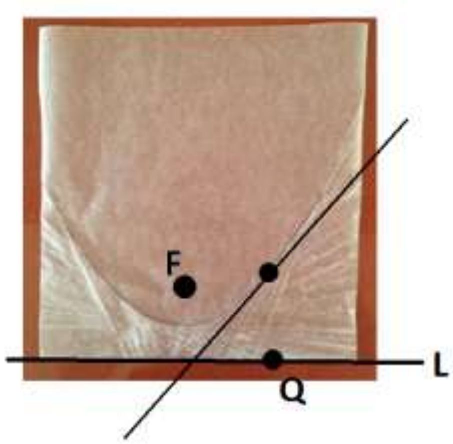
Here's another way to characterize regions I and II:

For a point P in the plane draw a circle with center P passing through F. This circle interests line L either 0, 1, or 2 times. Points in region I give circles that intersect 0 times (because $b < a$), points on the parabola give circles that intersect, actually just touch, 1 time (because $b = a$), and points in region II give circles that intersect 2 times (because $b > a$).

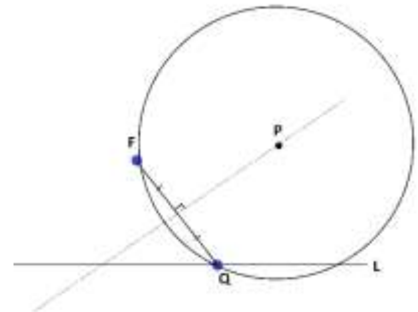
Crease lines on a folded sheet of paper:
Folding a point Q on the bottom edge L of a sheet of paper to a special interior point F produces a crease that is the perpendicular bisector of $\overline{QF}$.



We want to show that this crease line is a tangent line to the parabola with focus F and directrix L. That is, we need to show that there is precisely one point on this crease that sits on the parabola and that every other point on the crease lies in region II of this parabola.



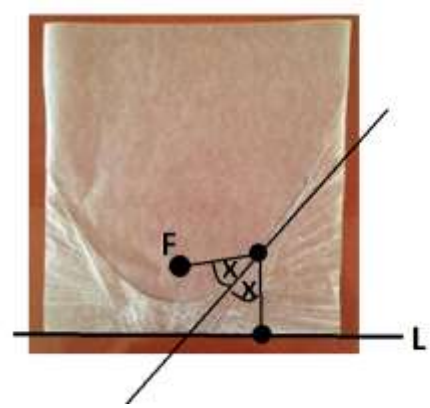
Being the perpendicular bisector, each point P on the crease is equidistant from F and Q. Thus if we draw a circle with center P passing through F, that circle also passes through Q. Consequently this circle intersects the bottom edge L either 1 or 2 times.



There is clearly precisely one point P on the crease with its circle tangent to L, that is, intersecting L just 1 time. (The point P with $\overline{PQ}$ perpendicular to L.) This special point is on the parabola. All other points on the crease give circles that intersect L twice and thus lie in region II.

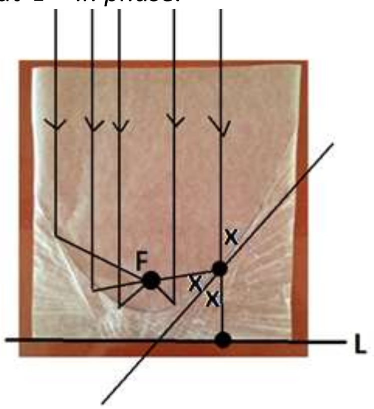
So each crease is indeed a tangent line to the parabola and all the folds do highlight region II and make region I and its boundary visible.

A BEAUTIFUL CONSEQUENCE
Folding produces congruent angles and so we have a proof of the reflection property of the parabola at hand.



(When a ray of light hits a curved mirror, it reflects off the mirror as though it were hitting the tangent line to the mirror at the point of contact. The angle of incidence matches the angle of reflection for the tangent line.)

REFLECTION PROPERTY: *Each ray of light perpendicular to the directrix* L *of a parabolic mirror reflects off the mirror to intercept the focus* F *. All such rays travel the same distance to reach* F *(the distance they would have traveled to* L *) and so arrive at* F *"in phase."*



(Look at vertical angles.)

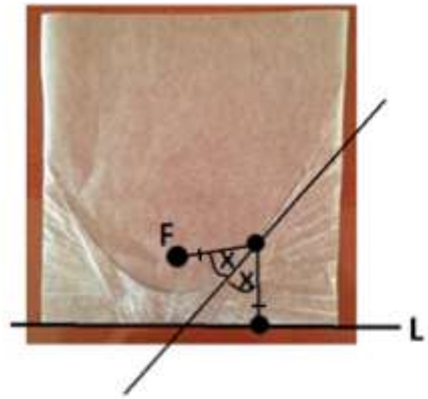
Reversing this: *Placing a light bulb at the focus of a parabolic mirror produces parallel rays of light.*

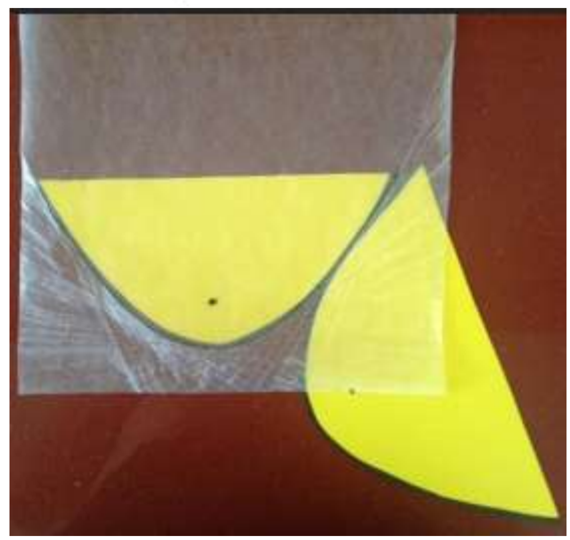
ROLLING A PARABOLA

Take two copies of the same parabola. Keeping one fixed in place, roll the other along its boundary. It is a bit hard to see in these photographs (sorry), but the focus of the rolling parabola moves along a perfect straight line!

As each still photo of the moving parabola is a picture of the fixed parabola and its reflection across a tangent line, this answers the opening puzzler – the locus of reflections is a straight line.

The following two pictures show that the path of the moving focus (or equivalently the locus of reflections of the fixed focus) is actually the directrx of the parabola.

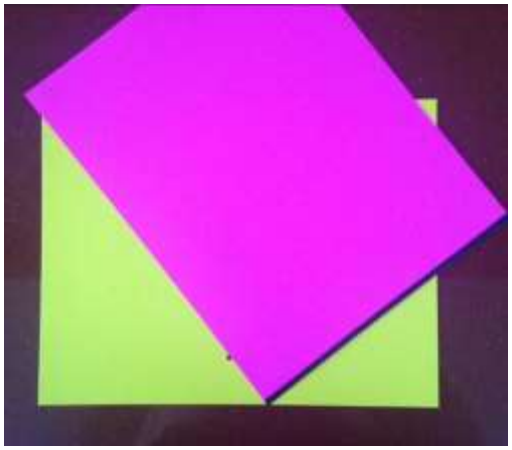


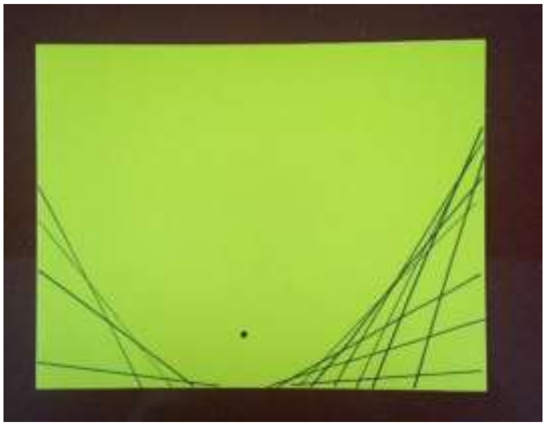


SQUARE CORNERS

Here's another way to create a parabola:

Again take a sheet of paper and mark a special point F *on it. Now lay the* 90° *corner of a second sheet on top so that one edge passes through* F *and the corner sits on the bottom edge of the first sheet. Trace the second edge of the corner. Repeat multiple times.*

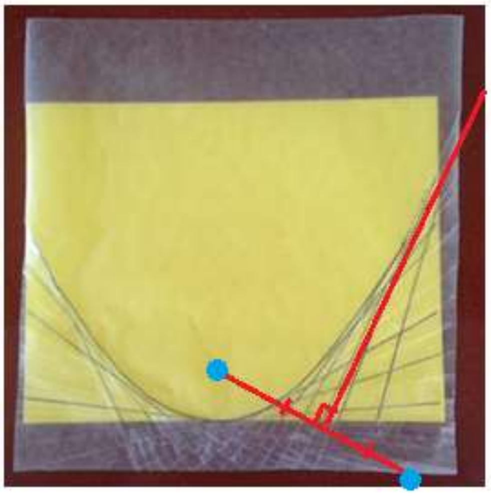




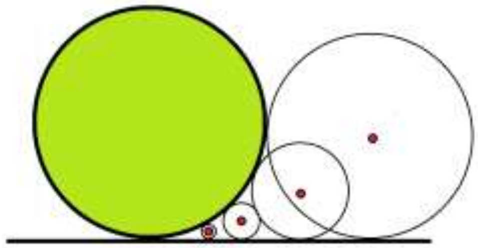
I followed this construction with a point F the same distance from the bottom edge as I had with my wax paper, and I see I get the very same parabola.

The directrix this time is below the bottom edge. In fact, it seems the directrix is the same distance below the bottom edge as the focus is from that edge.

This makes sense as the edge of 90° corner is the perpendicular bisector of a segment connecting F to a point on this lower line. It matches a crease in our first construction and so does give the same parabola.



ANOTHER PARABOLA



Given a fixed circle C and a tangent line L to the circle draw a set of circles each tangent to the two objects. Show that the centers of those circles lie on a parabola. (Where is the focus and where is the directrix of this parabola?)

FURTHER EXPLORATION:
What if L is bent to into a circle? Do the centers of the tangent circles trace the same type of conic curve? (There are two directions to bend L into a circle. Do they give the same results?)

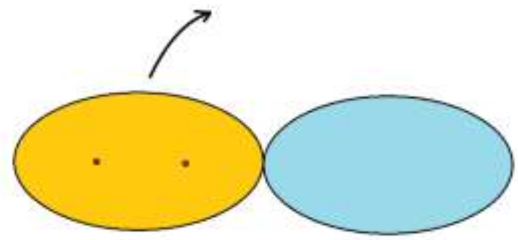
What if line L intersects the circle? What curves could the centers trace?

EVEN FURTHER EXPLORATION: We started this essay by folding a parabola as follows: Choose a fixed line L on a sheet of waxed paper (in our case, its bottom edge) and a fixed point F. Fold a point of L onto F and make the crease mark. Do this many times and the form of a parabola emerges.

What happens if L is a circle? Draw a large circle on a piece of wax paper and mark a point F inside it away from the center. Fold points of L onto F making crease marks each time. If you do enough folds all the way around the circle, what conic section emerges? (Can you prove any claim you make?) What reflection properties can you deduce?

Repeat this experiment, but this time have F be a point outside the circle.

EVEN MORE FURTHER EXPLORATION
Two congruent cardboard ellipses sit side-by-side just touching. If the left ellipse rolls around the edge of the right ellipse, fixed in place, what path does each focus trace?



STILL YET EVEN MORE FURTHER EXPLORATION
We also made a parabola by tracing one edge of a right-angle corner of a piece of paper.

What curves emerge if we use the edge of a piece of paper with a 60° corner instead? A 45° corner? Some arbitrary angle?

RESEARCH CORNER
As one parabola rolls along the other, what path does the vertex of the rolling parabola trace?

tanton.math@gmail.com