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WILDLY COOL MATH!

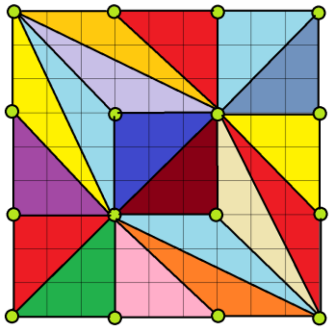
CURIOUS MATHEMATICS FOR FUN AND JOY

SEPTEMBER 2014

PUZZLER 1: Cover a rubber ball with dots, lots of dots. Color some dots azure (A), some dots blush (B), and some dots crimson (C). Now draw lots of non-intersecting lines connecting pairs of dots to make spherical triangles. Completely cover the surface with these triangles. (Make sure every surface region is bounded by three edges and three vertices.)

Juju looks at the design she made this way on a rubber ball and notices that one of her triangles has corners one of each color. Explain why Juju is sure to have at least one more ABC triangle on her ball.

PUZZLER 2: It is possible to divide a 9×9 grid of squares into eighteen triangles of equal area, each with a vertex at an intersection point.



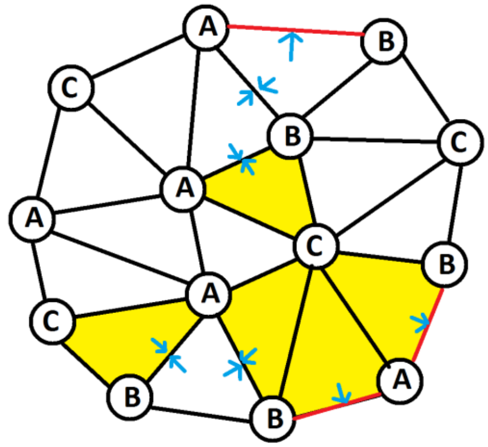
Is it possible to divide the 9×9 grid of squares into an odd number of triangles of equal area (with vertices at grid points)?

For each counting number N it is possible to divide an $N \times N$ grid of squares into an even number of triangles of equal area, each with vertices at intersection points. (Two triangles, for example!) Is there any value of N for which an $N \times N$ grid of squares can be so subdivided into an odd number of triangles?

SPERNER'S LEMMA

In 1928 German mathematician Emanuel Sperner developed the following delightful result:

Suppose a polygon is subdivided into triangles (making sure that each pair of neighboring triangles meet along an entire edge length). If we randomly label the vertices of all the triangles A, B, and C, then the count of outside edges labeled AB has the same parity as the count of triangles fully labeled ABC. (So if there are an odd number of outer AB edges, then there are an odd number of ABC triangles. If there are an even number of outer AB edges, then there are an even number of ABC triangles.)



This decagon has 3 outside AB edges and 5 interior ABC triangles. Both of these counts is odd.

The proof of this result is lovely too.

PROOF: Draw arrows inside each triangle pointing to any AB edges. Let's examine how many arrows there are.

<u>Counting by Triangles:</u> Every ABC triangle contains one arrow. All other triangles (CCC triangles, AAB triangles, ACC triangles, and so on) contain either zero or two arrows. It follows that the number of arrows has the same parity as the number of ABC triangles. (Thus if one count is odd or is even, the other is the same.)

<u>Counting by Edges:</u> Each outside AB edge has one arrow. Each inside AB edge has two arrows. All other edges have no arrows. So the number of arrows has the same parity as the number of outside AB edges.

We see that the number of arrows, the number of ABC triangles, and the number of outside AB edges all have the same parity.

Question 1: What can you say about the number of outside BC edges and the number of outside AC edges?

Question 2: Can you now prove that the number of ABC triangles in puzzle 1 has to be even?

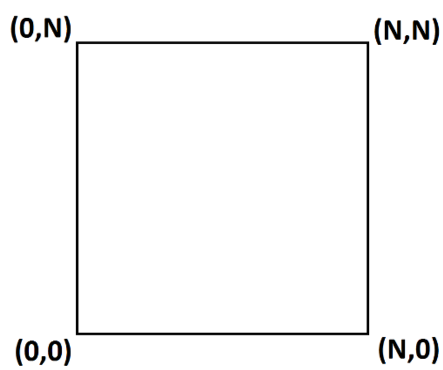
DIVIDING SQUARES INTO TRIANGLES OF EQUAL AREA

Let me give puzzle 2 away:

It is impossible to subdivide an $N \times N$ *grid of squares into an odd number of triangles of equal area (with vertices at grid points).*

Proving this in full generality is surprisingly hard!

Let's assume that our grid of squares is placed on a set of coordinate axes so that its vertices have coordinates $(0,0)$, $(N,0)$, (N,N), and $(0,N)$.



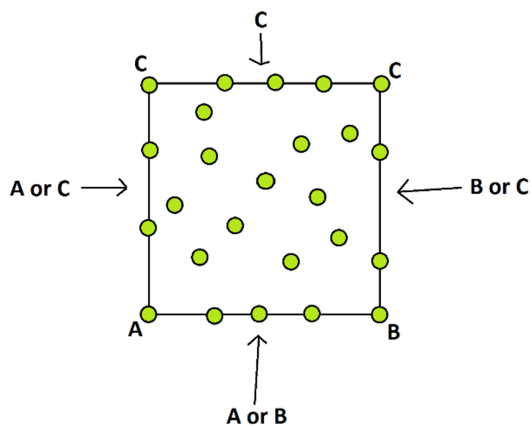
Suppose this square is divided into k triangles of equal area $\frac{N^2}{k}$.

Since the vertices of each triangle lie on grid points, each vertex has integer coordinates. Also, if a triangle has vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3), then its area is given by the shoelace formula:
$$\frac{1}{2}| x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1 |.$$
(See the 2014 COOL MATH ESSAY at http://www.jamestanton.com/?p=1072.) This area equals N^2/k for each triangle. Our job is to prove that k has to be even.

EASIER CASE: If N is an odd integer…

Let's give each triangle vertex (x, y) a label according to the following possibilities:

x even, y even:	Label it A.
x odd, y even:	Label it B.
y odd:	Label it C.



With this scheme, any vertex of the top edge (of the form (x, N)) has label C. The point $(0,0)$ has label A and any vertex on the left edge has label A or C. The point $(N,0)$ has label B, and all points on the bottom edge have label A or B. All vertices on the right edge have label B or C. Interior vertices can be A, B, or C.

Notice that all the outside AB edges of this polygon, divided into triangles, lie on the bottom edge. The number of these outside AB edges must be odd. (In reading from left to right along this bottom edge we start with A and end with B. Each AB edge "switches" the label and so there must be an odd number of switches.)

By Sperner's result, there is at least one ABC triangle in our diagram.

Suppose the vertex labeled A in this special triangle has coordinates (x_1, y_1) (both even), the vertex labeled B has coordinates (x_2, y_2) (x_2 odd, y_2 even), and the vertex labeled C coordinate (x_3, y_3) (y_3 odd).

We have:

$$\frac{N^2}{k} = \frac{1}{2}| x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1 |$$

Of the six terms on the right, all are even except x_2y_3. Thus the right side is a fraction of the form: $\frac{odd}{2}$. Since N is odd, it better be the case that k is even.

HARDER CASE: If N is an even integer…

If N is even we can write $N = 2^a m$ for some $a \ge 1$ and some odd integer m.

We'll now follow the same proof as before, but we'll be more explicit about the powers of two that appear in all our integers.

Give each triangle vertex (x, y) a label as follows:

Write $x = 2^b p$ with p odd and $b \ge 0$, and $y = 2^c q$ with q odd and $c \ge 0$. (Or if $x = 0$, a number highly divisible by two, declare $b = \infty$, an infinitely large number, and interpret the role of x appropriately in the remainder of this proof. Ditto if $y = 0$.)

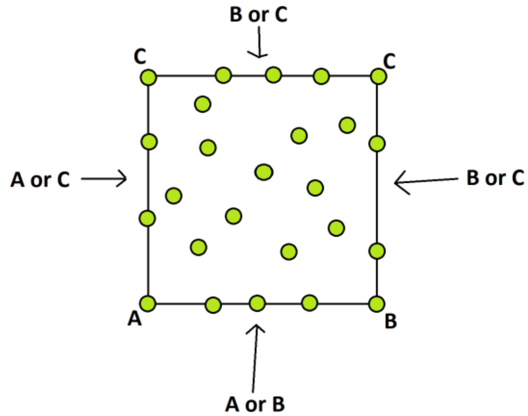
Label (x, y):

A *if $b > a$ and $c > a$.*
B *if at least one of b or c is $\le a$, and $b < c$.*
C *if at least one of b or c is $\le a$, and $b \ge c$.*

NOTE: For the case with N an odd number we have $a = 0$, and these labels match the labels we assigned before.

Also note that the condition for B forces $a \ge b$ and the condition for C forces $a \ge c$.

The vertices that lie on the corners and the edges of the square will have labels almost the same as before:



We see again that there must be an odd count of outside AB edges, and so our diagram contains at least one ABC triangle.

Suppose the vertex labeled A in this triangle has coordinates (x_1, y_1), the vertex labeled B has coordinates (x_2, y_2), and the vertex labeled C coordinate (x_3, y_3). (Notice that x_2 cannot be zero, and y_3 cannot be zero.) We have:

$$\frac{N^2}{k} = \frac{1}{2}| x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1 |$$

That is,

$$| x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1 |$$

is an integer of the form $\frac{2^{2a+1}m^2}{k}$ with m odd. If k is odd, then this integer is divisible by 2^{2a+1}. We'll show that this is not possible, forcing us to conclude that k is even.

Writing $x_1 = 2^{b_1} p_1$, $y_3 = 2^{c_3} q_3$, and so on, we have:

$$\begin{array}{ccc} b_1 > a & c_2 > b_2 & b_3 \ge c_3 \\ c_1 > a & a \ge b_2 & a \ge c_3 \end{array}$$

Now:

$$x_1y_2 = 2^{b_1+c_2}\, p_1q_2 > 2^{a+b_2}\, p_1q_2 \ge 2^{c_3+b_2}\, p_1q_2$$
$$x_2y_3 = 2^{b_2+c_3}\, p_2q_3$$
$$x_3y_1 = 2^{b_3+c_1}\, p_3q_1 \ge 2^{c_3+a}\, p_3q_1 \ge 2^{c_3+b_2}\, p_3q_1$$
$$y_1x_2 = 2^{b_2+c_1}\, p_2q_1 > 2^{b_2+c_3}\, p_2q_1$$
$$y_2x_3 = 2^{b_3+c_2}\, p_3q_2 > 2^{b_2+c_3}\, p_3q_2$$
$$y_3x_1 = 2^{b_1+c_3}\, p_1q_3 > 2^{b_2+c_3}\, p_1q_3$$

We have that each term has $2^{b_2+c_3}$ as a factor, with this being that largest power of two that divides x_2y_3. This means that $|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1|$ is an integer divisible by highest power of two $2^{b_2+c_3}$:

$$|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1| = 2^{b_2+c_3} \times odd$$

But $2^{b_2+c_3} \le 2^{a+a} = 2^{2a}$. This integer can't be of the form $\frac{2^{2a+1}m^2}{k}$ if k is odd.

That's it!

VERY HARD CHALLENGE: *Prove that it is impossible to divide a square of any size into an odd number of triangles of the same area. (The vertices of the triangles can now lie anywhere within the square: they need not have integer coordinates!)*

Comment: Our proof for puzzle 2 makes use of the highest powers of two that divide various integers. By using a "2-adic valuation" one can extend our approach to establish the claim made in the very hard challenge! (Or can you come up with a simple proof that does not make use of advanced mathematical tools? If you do, let me know!)

RESEARCH CORNER:

Show that one can divide a 9×9 square into 2, 6, 18, 54, and 162 triangles of equal area and with vertices having integer coordinates. Are any other counts of triangles possible?

For each N what are the possible counts of triangles of equal area, with vertices at integer points, subdividing an $N\times N$ grid of squares?

MORE RESEARCH:
Care to subdivide an equilateral triangle into triangles of equal area? Into quadrilaterals of equal area? Cubes into tetrahedra of equal volume?
(Oh heavens!)

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