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Uplifting Mathematics for All

WHOA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

OCTOBER 2020

THIS MONTHS' PUZZLER:

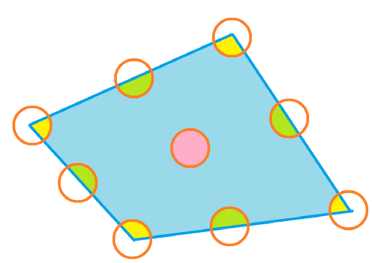
Draw a polygon in the plane.

At each corner draw a small circle centered at the corner and color yellow the interior of the circle that lies within the interior of the polygon.

At each edge, draw a small circle with center on the edge. Color green the interior of the circle that lies within the interior of the polygon.

Within the interior of the polygon draw a small circle and color its area pink.

Assume all the circles are congruent and are quite small compared to the size of the polygon so that the only edges that enter a circle are ones that meet its center.



Prove that the sum of values of the yellow areas minus the sum of the values of the green areas plus the value of the pink area equals zero.

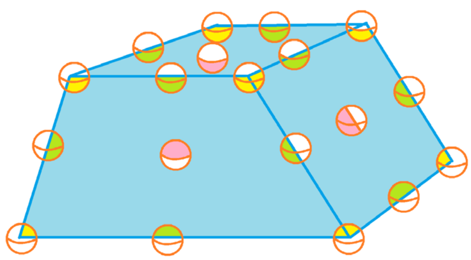
YELLOW- GREEN + PINK = 0

Going Further:

What is the one-dimensional version of this claim—and is it true?

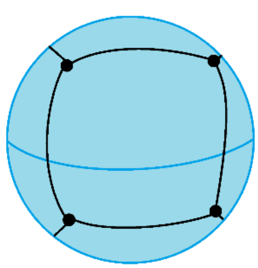
Going Further Still:

What is the three-dimensional version of this claim—and is it true?



EULER'S FAMOUS POLYHEDRAL FORMULA

Imagine a hollow cube with edges and faces made of rubber. If one pumps air into the cube, the edges and faces can bend and expand and the whole cube becomes a sphere.



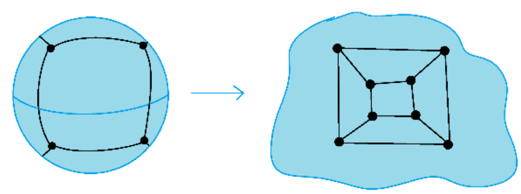
This means that we can think of a cube as a figure drawn on the surface of a sphere. It has 8 dots (*vertices*) drawn on the surface, with 12 *edges* connecting certain pairs of dots, dividing the surface into 6 regions (*faces*).

Notice that $6-12+8=2$.

For any polyhedron with V vertices, E edges, and F faces equivalent to a figure drawn in the surface of a sphere, Swiss mathematicians Leonhard Euler observed that the equation $V-E+F=2$ holds.

Challenge: *Draw an example of a polyhedron that does not expand to "become a sphere" when pumped with air. If you can draw such a solid, what value does it give for* $V-E+F$ *?*

This famous result can be interpreted as a result about diagrams drawn in the plane, rather than on the surface of a sphere: Simply puncture the sphere at some point in the interior of a face and flatten out the stretchy rubber of the sphere into a flat region.

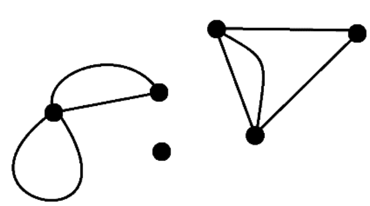


The general version of this result attributed to Euler is this.

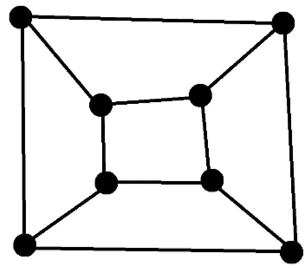
Suppose V *vertices are drawn on a plane and there are* E *edges connecting pairs of vertices (or perhaps an edge starts and ends at the same vertex), and that these edges divide the plane into* F *regions of finite area. If the diagram comes in* C *connected components, then*

$$V-E+F=C.$$

For example, the next picture with $V=6$ vertices, $E=7$ edges, $F=4$ regions of finite area is composed of $C=3$ connected pieces, and indeed $6-7+4=3$.



For a diagram arising from a polyhedron drawn on the surface of the sphere, we're sure to have $C=1$, and so $V-E+F=1$.



For the cube $V=8$, $E=12$, $F=5$, *and* $C=1$.

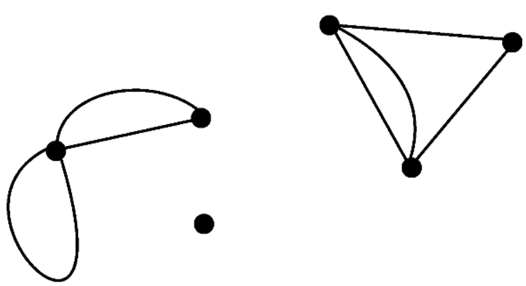
But here F is counting only the planar regions of finite area. The outer region of infinite area comes from a face of the polyhedron, and so the count of polyhedral faces is $F+1$. We have

$$V-E+(F+1)=2,$$

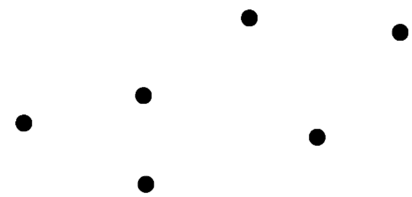
which is the formula for polyhedra.

ESTABLISHING THE FORMULA

Here's an intuitive way to establish that $V-E+F=C$ for any diagram of vertices and edges drawn in the plane.



Start by removing all the edges to leave a diagram of V isolated dots.



This stripped diagram has V vertices, $E=0$ edges, and $F=0$ regions of finite area and the diagram comes in $C=V$ components. The formula $V-E+F=C$ holds.

Now draw back in one edge.

If it connects two vertices, then we've reduced the count of components by one.

$$E \to E+1$$
$$C \to C-1$$

If we've drawn an edge from a vertex back to itself, then we have created a finite region, but have not changed the count of components.

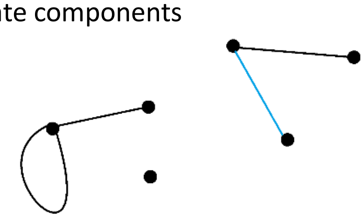
$$E \to E+1$$
$$F \to F+1$$

Either way, the equation

$$V-E+F=C$$

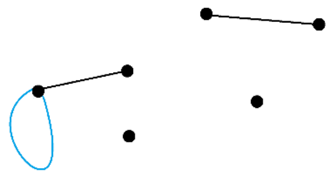
remains balanced and true.

In general, adding back in an edge to an existing diagram either connects two separate components



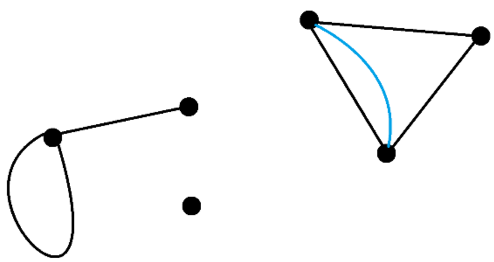
$$E \to E+1$$
$$C \to C-1$$

or connects a vertex to itself,



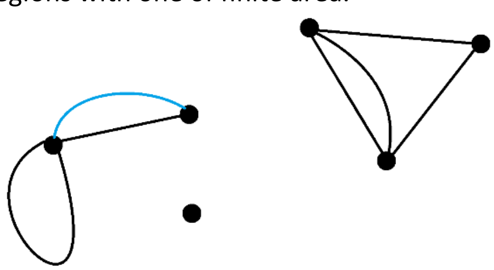
$$E \to E+1$$
$$F \to F+1$$

or splits an existing region of finite area into two regions of finite area,



$$E \to E+1$$
$$F \to F+1$$

or splits the region of infinite area into two regions with one of finite area.



$$E \to E+1$$
$$F \to F+1$$

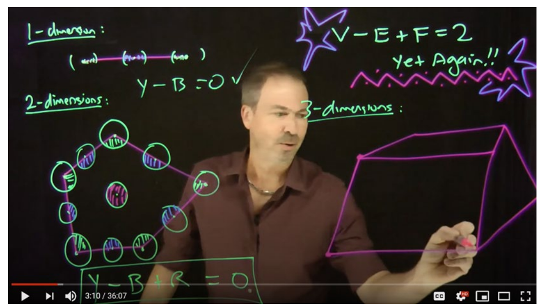
In all cases, the return of an edge to the picture keeps the equation $V - E + F = C$ balanced and true, and so this equation must be true for the original diagram.

Comment: For a more robust—and historically tied—proof of the original polyhedral formula, see this essay by Abigail Kirk

THE OPENING PUZZLER

I've recently been playing with the sections of volumes of spheres in whacky ways and stumbled upon the result of the opening puzzle – the three-dimensional version, that is. It turns out to be an utterly surprising re-interpretation of $V - E + F = 2$ for polyhedra. It left me gob smacked.

That YELLOW – GREEN + PINK – BLUE = 0 holds in three (and higher) dimensions turns out to be known. But when I looked at the literature, I found the proofs to be quite opaque. So I made this video of my approach to it.



The story presented there takes us down the path of understanding the surface area of polygons drawn on spheres. The only point I did not prove in the video is that $V - E + F = 2$ for spherical polyhedra. But we've just completed that story now!

This essay really an invitation to watch the video.

RESEARCH CORNER

What is Euler's formula for "four-dimensional polyhedral" and what is the four-dimensional version of the opening puzzler?

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