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WILD COOL MATH!

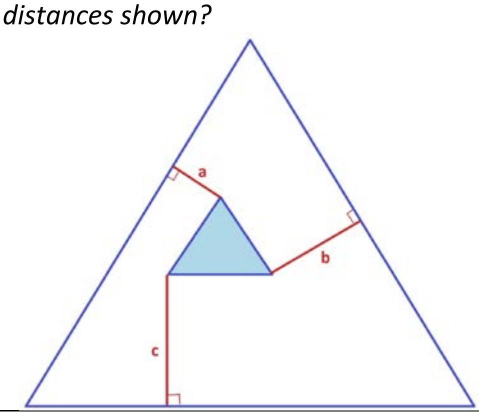
CURIOUS MATHEMATICS FOR FUN AND JOY

October 2018

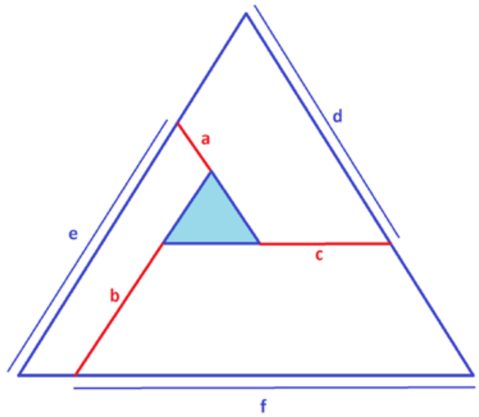
THIS MONTH'S PUZZLERS

I recently had some fun on Twitter sharing a whole series of puzzles related to Viviani's famous theorem. I mention this theorem in my January 2017 essay and explore it in detail in the February 2017 piece. For this essay, I am sharing some of those twitter puzzles and going through their solutions. (And don't worry! This essay is self-contained so there is no need to look up either of those previous pieces.)

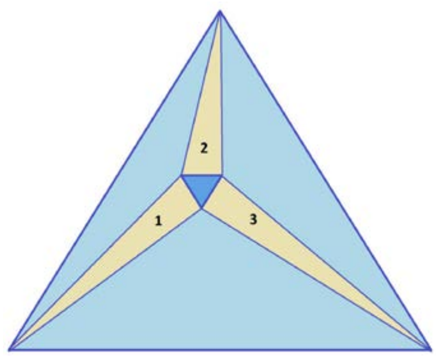
PUZZLE 1: *An equilateral triangle moves around inside a large equilateral while maintaining its orientation, always matching that of the large one. What can you say about the sum* $a+b+c$ *of distances shown?*



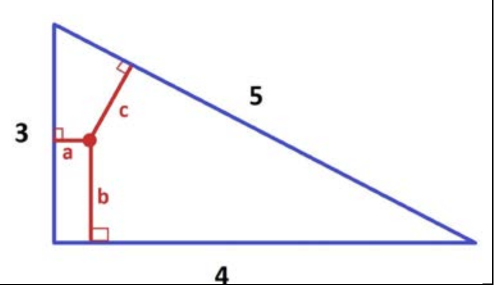
PUZZLE 2: *An equilateral triangle moves around inside a large equilateral while maintaining its orientation to always match that of the large one. What can you say about the sum* $a+b+c$ *of distances shown? What about the sum* $d+e+f$ *?*



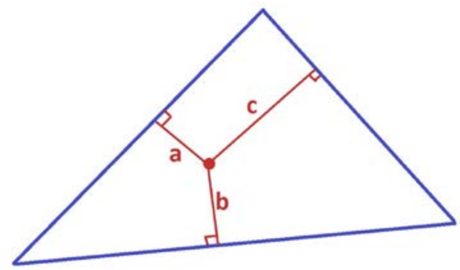
PUZZLE 3: *An inverted equilateral triangle moves around inside a large equilateral while maintaining its inverted orientation. What can you say about the sum of areas 1, 2, and 3 shown?*



PUZZLE 4: *For which points inside a 3-4-5 right triangle does the sum* $3a+4b+5c$ *for distances* a, b, c *shown have the value* 12*?*

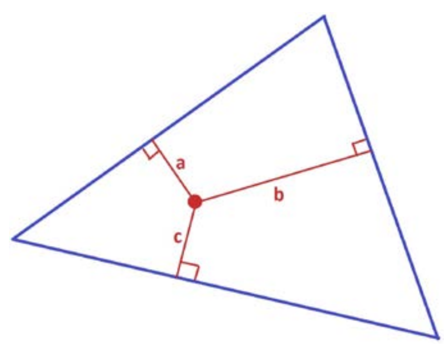


PUZZLE 5: *A triangle has the property that for each point inside the triangle the value of the sum* $a+b+c$ *of distances shown is the same fixed value. Must the triangle be equilateral?*



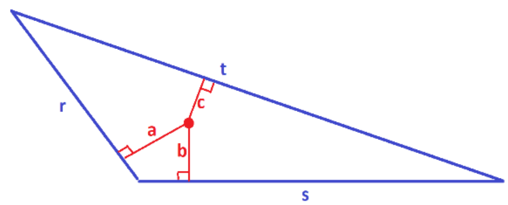
What if, instead, $a+b+2c$ *has the same fixed value? What can you say about the triangle?*

PUZZLE 6: *A triangle has the property that for three fixed values* r, s, *and* t, *the combination* $ra+sb+tc$ *of distances shown is the same for each point inside the triangle. What can you say about the triangle?*



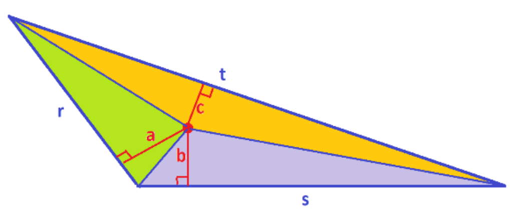
SOME RESULTS ABOUT TRIANGLES

Consider a triangle with side lengths r, s, and t, and consider the three distances a, b, and c shown for a given point inside the triangle. These are its distances from the three sides of the triangle.



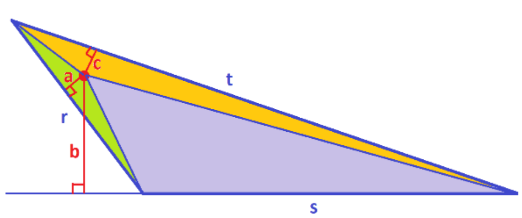
Then I claim that the combination $ra+sb+tc$ is sure to equal twice the area of the triangle. (And it follows then that the combination $ra+sb+tc$ is fixed in value for all points inside the triangle.)

To see this, subdivide the triangle into three triangles as shown.



The sum of areas of these triangles is $\frac{1}{2}ra+\frac{1}{2}sb+\frac{1}{2}tc$ and this equals the area of the whole triangle. The result follows.

This result is still true even if we must extend the side of triangle to measure a distance.



Challenge: *What if the chosen point wanders outside the triangle? Describe the region of the plane where the combination $ra+sb-tc$ is constant in value. Describe the region where $-ra-sb+tc$ is constant in value.*

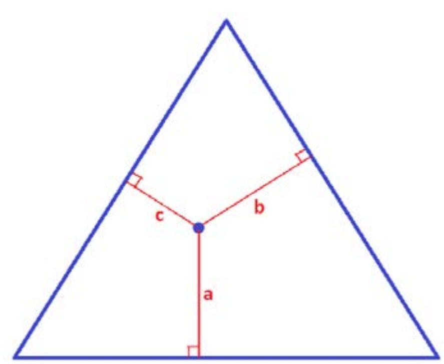
PUZZLE 4 SOLUTION: This work shows that for the 3-4-5 triangle, $3a+4b+5c$ equals twice the area of the triangle, which is 12, for *all* points inside the triangle.

If $r=s=t$ then we are talking about an equilateral triangle with side length r. It follows that $ra+rb+rc=2A$, where A is the area the triangle, and so

$$a+b+c$$

is constant in value for all points inside the triangle. One can check that $\frac{2A}{r}$ equals the height h of the equilateral triangle. (Or be clever and note that if the point lies at the apex of the triangle, then the sum $a+b+c$ equals $0+0+h$. Since the sum is known to be constant, that constant must thus be h.) We have …

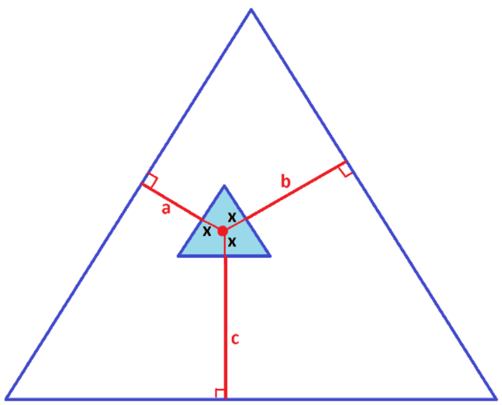
Viviani's Theorem: *For each point inside an equilateral triangle, the sum $a+b+c$ of distances shown is fixed in value (and equals the height of the triangle).*



We're now also fully equipped to solve the puzzlers.

PUZZLE 1 SOLUTION

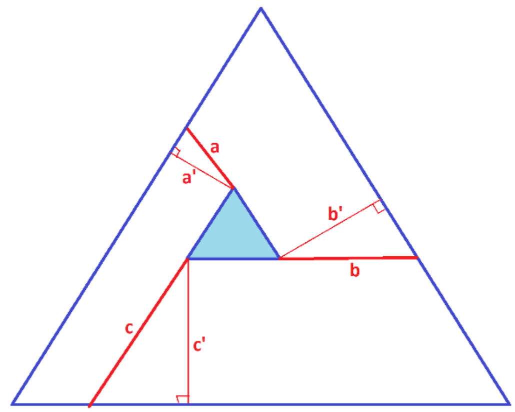
Shift the line segments a tad so that they align with the center of the small equilateral triangle. Notice that the distance x shown does not change as the small triangle moves.



We have that $(a+x)+(b+x)+(c+x)$ is fixed in value and equals the height of the large triangle, and $3x$ is the height of the small triangle (by Vivian's theorem too!). Thus $a+b+c$ is fixed in value and equals the difference of the two triangle heights.

PUZZLE 2 SOLUTION

If you know your 30-60-90 triangle ratios, then we see in the next figure that we have $a=\frac{2}{\sqrt{3}}a'$, $b=\frac{2}{\sqrt{3}}b'$, and $c=\frac{2}{\sqrt{3}}c'$.
Since $a'+b'+c'$ is fixed in value, it follows that $a+b+c$ is too. (By the way, I don't know my ratios, but I do see three similar triangles and so I can say $a=ka'$, $b=kb'$, $c=kc'$ for some value k and that is all I need.)



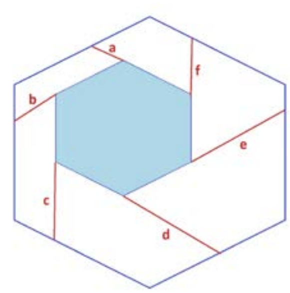
Pushing the small triangle up to the apex of the large one we see that $a+b+c$ equals $0+0+(S-s)$ where S is the side-length of the large triangle and s is the side-length of the small one.

And looking at the original diagram we see that

$$\begin{aligned} d+e+f &= (S-a)+(S-b)+(S-c) \\ &= 3S-(S-s) \\ &= 2S+s. \end{aligned}$$

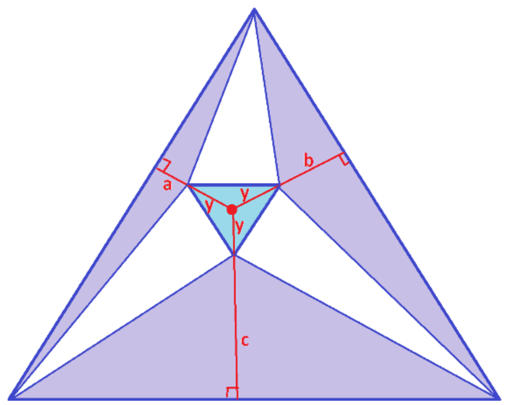
(We also see this with the small triangle at the apex of the large.)

Challenge: *A regular hexagon moves around inside a large regular hexagon while maintaining its orientation to always match that of the large one. What can you say about the sum of distances shown?*



PUZZLE 3 SOLUTION

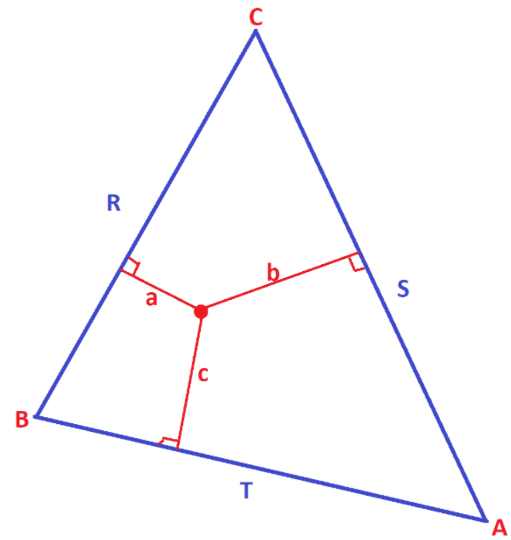
Draw the center of the inverted triangle and the three red line segments as shown.



Consider the distances shown and note that the value of y does not change as the small triangle moves. We know that the sum $(a+y)+(b+y)+(c+y)$ is fixed in value, and so it follows that the sum $a+b+c$ is fixed in value too. It then follows that the sum of the three purple areas is fixed. As the area of the small triangle is also fixed, we have that the sum of white areas is fixed too. This is the sum of areas we were asked to consider.

PUZZLE 5 and 6 SOLUTIONS

Suppose a triangle, ΔABC, with side lengths R, S, and T has the property that, for some fixed values r, s, and t, the combination $ra+sb+tc$ of distances shown is the same for all points inside the triangle. Our challenge is to say something about the values of R, S, and T.



Placing the point at each vertex of the triangle shows that we must have

$$rH_A = sH_B = tH_C$$

where H_A is the height of the triangle as measured from vertex A, and so on. Now the area of the triangle is given by

$$area = \frac{1}{2}RH_A = \frac{1}{2}SH_B = \frac{1}{2}TH_c$$

and so $H_A = \dfrac{2 \times area}{R}$, and so on.

From $rH_A = sH_B$ it now follows that $\dfrac{r}{R} = \dfrac{s}{S}$, or $\dfrac{R}{S} = \dfrac{r}{s}$ and so we have a pair of equivalent fractions. Thus $R = kr$ and $S = ks$ for some number k.

Similarly, from $sH_B = tH_C$ we get more equivalent fractions: $\dfrac{S}{T} = \dfrac{s}{t}$. Since $S = ks$ it follows that $T = kt$ as well.

This means that our triangle with side lengths R, S, and T is just a scaled version of a triangle with side lengths r, s, and t. This answers puzzle 6.

To answer the first part of puzzle 5:

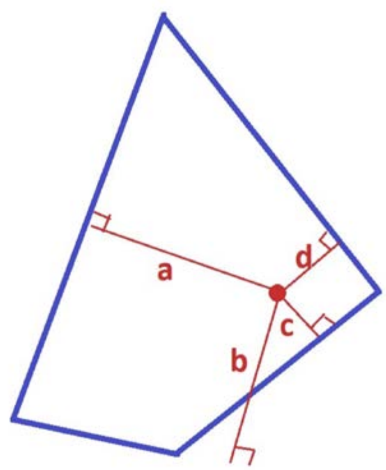
Here we have $r = s = t = 1$ and so the triangle must be a scaled version of an equilateral triangle. That is, it must indeed be an equilateral triangle.

To answer the second part of puzzle 5:

Here we have $r = s = 1$ and $t = 2$. Since no triangles of these side lengths exists (except for a generic one) there are no triangles for which $a + b + 2c$ is constant for all points inside the triangle.

RESEARCH CORNER

A quadrilateral has the property that for each point inside the quadrilateral the sum of its distances from each of the four sides (perhaps extending a side length to compute this) is the same fixed value. Must the figure be a parallelogram?



a + b + c + d = constant fixed value

I personally don't know how to prove this! Is it true? If so, how might the proof proceed?

More generally: If a convex polygon has the property that for each point inside the polygon the sum of its distances from each side of polygon has the same fixed value, must the polygon possess rotational symmetry?

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