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WHOA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

October 2016

THIS MONTH'S PUZZLER
It's a classic one.

The number 200 *can be written as the sum of two or more consecutive positive integers two different ways*

$$\begin{aligned} 20 &= 5+6+7+\cdots+19+20 \\ &= 38+39+40+41+42 \end{aligned}$$

Is there a third way to write 200 *as such a sum?*

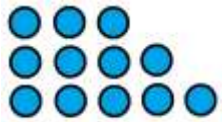
In how many different ways can the number 5000 *be written as a sum of two or more consecutive positive integers?*

Which numbers cannot be written as a sum of two or more consecutive integers?

Which numbers cannot be written as a sum of three of more consecutive integers?

TRAPEZOIDAL NUMBERS

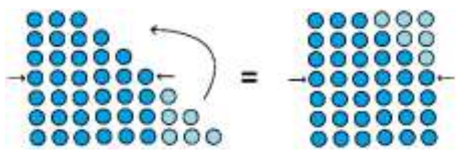
A number is called *trapezoidal* if it can be written as a sum of two or more consecutive positive integers. For instance, 12 is trapezoidal as $12 = 3+4+5$. Moreover, this presentation shows how to arrange 12 pebbles into a trapezoid with increasing row length. (We declare that a trapezoid must have at least two rows.)



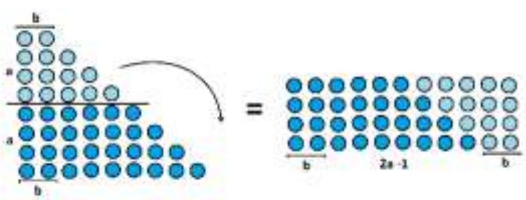
$12 = 3 + 4 + 5$

Any trapezoidal arrangement of pebbles can be turned into a rectangular array with an odd number of rows. Here's how.

If the trapezoid itself has an odd number of rows, simply identify the middle row and swing a triangular configuration of pebbles from below to top as indicated. This gives a rectangle with the same odd number of rows.



If the trapezoid has an even number of rows, swing the top set of rows 180° and adjoin them to the bottom set of rows as shown. This gives a rectangle with an odd number of columns.



Flipping the diagram gives a rectangle with an odd number of rows.

We see now that **every trapezoidal number has a non-trivial odd factor.**

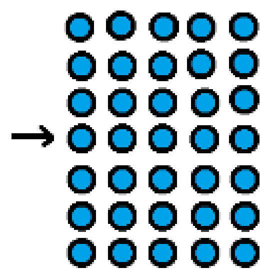
As the powers of two $1, 2, 4, 8, 16, \ldots$ fail to have odd factors greater than one, no power of two is trapezoidal.

Are these the only non-trapezoidal numbers? If $N = m \times n$ with m an odd integer greater than one, is N sure to be trapezoidal?

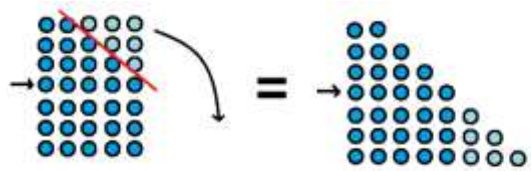
The answer is yes!

FROM RECTANGLES TO TRAPEZOIDS

Let $N = m \times n$ with m a non-trivial odd number. Consider a rectangular array of dots with m rows and n columns. We can identify a middle row.

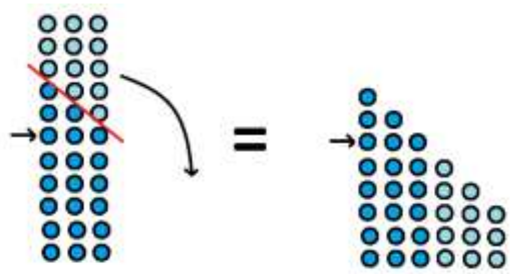


Draw a diagonal line as shown just after the rightmost dot of the middle row.

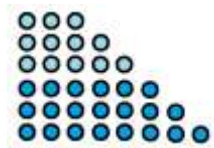


If the rectangle is "generous" enough, that is, wide enough, this identifies a triangular group of dots which we can rotate 180° to obtain a trapezoidal array, again with m rows. (This situation occurs when $\frac{m-1}{2} < n$.) We see N as a sum of an odd number of consecutive integers

If, on the other hand, the rectangle is somewhat "skinny" (that is, if $\frac{m-1}{2} \geq n$), then, we obtain a rotated trapezoidal diagram with $2n$ columns.



Flipping the image gives a trapezoidal diagram arranged by rows.



Here we see N as a sum of an even number of consecutive integers.

We have that **each non-trivial odd factor of a number N leads to a trapezoidal representation of N.** Thus every non-power of two can be written as a sum of two or more consecutive integers.

The set of all numbers that can be written as a sum of two or more consecutive positive integers is precisely the set of all non-powers of two.

We can go a little further. Call a factorization $N = m \times n$, with m a non-trivial odd factor, "generous" if $\frac{m-1}{2} < n$ and "skinny" otherwise. (And let's describe a $m \times n$ rectangle of dots by the same terms.) We have actually shown that if N has a generous factorization, then N can be written a sum of an odd number of consecutive integers, and if N has a skinny representation, then it can be written as an even number of consecutive integers.

TWO TRANSFORMATIONS

We have now two transformations of pictures of dots. We can take a trapezoidal array of dots and convert it to a rectangular array with an odd number of rows. (Let's call this transformation R for "turn it into a rectangle.") And we can take a rectangular array of dots with an odd number of rows, and convert into a trapezoidal array of dots. (Let's call this transformation T for "turn it into a trapezoid.") Recall we are assuming that no picture is allowed to have just one row.

Are these inverse transformations?

If we start with a trapezoid and convert it to a rectangle via R and then convert it back to a trapezoid via T, are we back to the trapezoid we started with?

How about starting with a rectangle? Does $R\left(T\left(\text{rectangle}\right)\right) = \text{ original rectangle}$?

Actually, we have two types of rectangles to consider: generous ones and skinny ones. And we have two types of trapezoidal arrays: ones with an odd number of rows (call them "odd trapezoids") and ones with an even number of rows ("even trapezoids").

Looking back at over diagrams we see

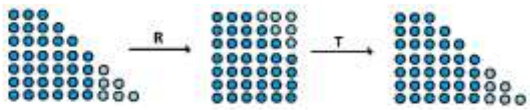
$$R\left(\text{odd trapezoid}\right) = \text{a generous rectangle}$$
$$T\left(\text{generous rectangle}\right) = \text{an odd trapezoid}$$

and

$$R\left(\text{even trapezoid}\right) = \text{a skinny rectangle}$$
$$T\left(\text{skinny rectangle}\right) = \text{an even trapezoid.}$$

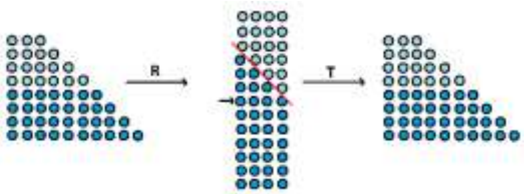
So if R and T are inverse transformations, they are so within the generous rectangle/odd trapezoid and skinny rectangle/even trapezoid sets.

Can we see that R and T are inverses? Yes!

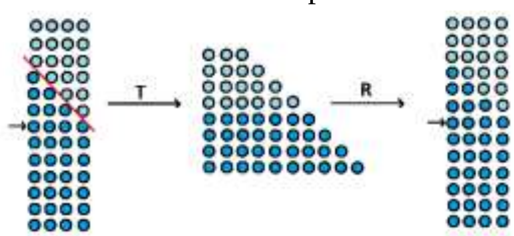


$$T\left(R\left(\text{odd trapezoid}\right)\right)$$
$$= \text{same odd trapezoid}$$

$$R\left(T\left(\text{generous rectangle}\right)\right)$$
$$= \text{same generous rectangle}$$



$$T\left(R\left(\text{even trapezoid}\right)\right)$$
$$= \text{same even trapezoid}$$



$$R\left(T\left(\text{skinny rectangle}\right)\right)$$
$$= \text{same skinny rectangle}$$

So R and T do indeed undo each other.

COUNTING PRESENTATIONS

The number 200 has two non-trivial odd factors, 5 and 25, giving a 5×40 rectangle (generous) and a 25×8 rectangle (skinny). The first leads to the odd trapezoidal representation $38+39+40+41+42$ and the second to the even trapezoidal representation $5+6+\cdots+20$.

Could there be a third trapezoidal representation of the number 200?

Let's answer this question in a general setting.

Suppose we are given a positive integer N.

Let's regard each non-trivial odd factor m of N as a picture of an $m\times n$ array of dots (here $n=\frac{N}{m}$) with an odd number of rows (namely, m rows), and regard each way of writing N as a sum of consecutive integers as a picture of a trapezoid. With this context writing

$T\left("5\times 40"\right) = "38+39+40+41+42"$

and

$R\left("5+6+\cdots+20"\right) = "25\times 8"$, for example, makes sense.

Let's now focus on the transformation T as a map from the set of all non-trivial odd factors of N to the set of all representations of N as sums of consecutive integers. We know that R is an inverse transformation for this map.

This map is one-to-one

Could two different non-trivial odd factors of N, m_1 and m_2, lead to the same trapezoidal sums $T\left("m_1\times n_1"\right)$ and $T\left("m_2\times n_2"\right)$?

No!

If $T\left("m_1\times n_1"\right) = T\left("m_2\times n_2"\right)$, then $R\left(T\left("m_1\times n_1"\right)\right) = R\left(T\left("m_2\times n_2"\right)\right)$, giving $"m_1\times n_1" = "m_2\times n_2"$, showing that m_1 and m_2 were the same odd factors after all.

This map is onto

Could there be sum S that fails to appear from applying the map T ?

No!

We can certainly construct the rectangle $R(S)$ from the sum. And then the image of this rectangle under T is $T(R(S)) = S$.

So we have just learned that our map T , which takes each non-trivial odd factor of N and creates from it a rectangle and then a trapezoid, produces each and every possible representation of N as a sum of consecutive integers, and without overlap.

We have

> **The number of different ways to write N as a sum of two or more consecutive positive integers matches the count of non-trivial odd factors of N .**

As 200 only has two non-trivial odd factors, there are indeed only two ways to write it as a sum of two or more consecutive terms.

Let's push things a little deeper.

Each non-trivial odd number $2k+1$ has at least one non-trivial odd factor, namely itself, and so has at least one trapezoidal representation. We can see that $(k)+(k+1)$ is such a representation.

> **No prime number is the sum of three or more consecutive positive integers.**

Each odd prime has just one non-trivial odd factor, and so has just one possible presentation. It must be the one of the form $(k)+(k+1)$ with just two terms.

(The even prime 2 is a power of two and has no trapezoidal presentation.)

On the other hand, each odd composite number $2k+1$ has at least two non-trivial odd factors. As $(k)+(k+1)$ is the only trapezoidal presentation with two terms, one of the factors must lead to a presentation with three or more terms.

> **The set of all numbers that can be written as a sum of three or more consecutive positive integers is precisely the set of all non-primes and non-powers of two.**

Final comment: We have the goods in this essay to make a stronger assertion. The number of different ways to write N as a sum of a (non-trivial) odd number of consecutive terms matches the number of generous odd factors of N , and the number of ways to write N as a sum of an even number of terms matches the count of skinny odd factors of N .

For example, as 200 has precisely one generous odd factor and precisely one skinny odd factor, and it has precisely one trapezoidal presentation with an odd number of terms and precisely one with an even number of terms.

RESEARCH CORNER

Take your favorite arithmetic sequence. Classify those numbers that can be expressed as a sum of two or more consecutive terms of your sequence. (Three or more? Four or more?)

Classify those numbers that are the sum of two or more consecutive triangle numbers, or square numbers, or Fibonacci numbers.

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