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WHAT COOL MATH!

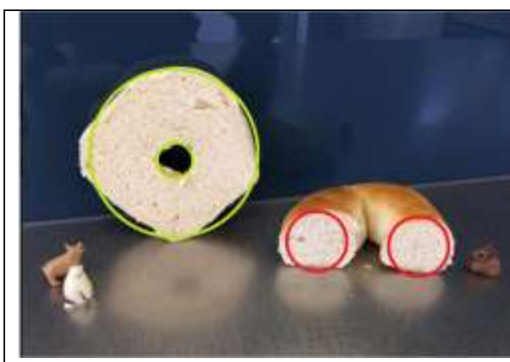
CURIOUS MATHEMATICS FOR FUN AND JOY

November 2016

THIS MONTH'S PUZZLER

There are two standard ways to slice a bagel: along a line of latitude in preparation for spreading cream cheese on the bagel, and along a line of "longitude," in preparation for eating only half.

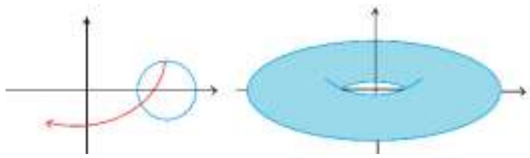
Both planar slices produce, for ideal toroidal bagels, two perfect circles in the cross-section shape.



Is there a third way to slice a bagel with a planar cut so as to produce two perfect boundary circles on the cross section?

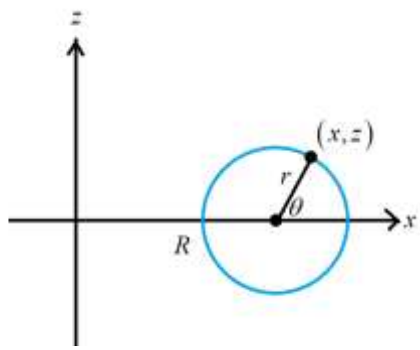
THE EQUATION OF A TORUS

One obtains a torus (a "perfect" donut surface) by rotating a circle about an axis.



Let's work with a circle of radius r centered on the x-axis. Suppose the center is R units away from the origin with $R > r$. Call the vertical axis the z-axis and assume that the y-axis is pointing out of the page towards us.

Any point on the circle in the xz-plane is determined by a single angle θ of measure between 0° and 360°.

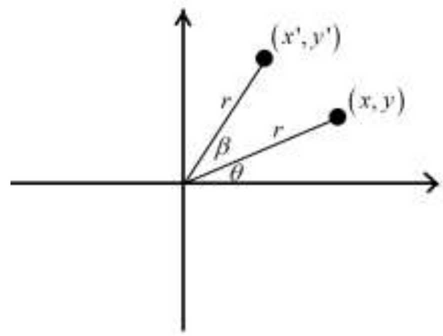


The diagram shows that its xz-coordinates are given by

$$x = R + r\cos\theta$$
$$z = r\sin\theta.$$

Let's now rotate this point about the z-axis through some angle ϕ, also between 0° and 360° in measure. This will give the general formulae for the coordinates (x, y, z) of a point on a torus. These formulae will be in terms of θ and ϕ.

An Aside on Rotations: Suppose (x, y) is a point in the plane and we rotate that point through an angle β about the origin. What are the coordinates of the image point (x', y') in terms of x and y?



To answer this write the coordinates (x, y) in polar form. Then, if

$$x = r\cos\theta$$
$$r = r\sin\theta,$$

we have

$$x' = r\cos(\theta + \beta)$$
$$y' = r\sin(\theta + \beta).$$

The trigonometric addition formulae give

$$\begin{aligned} x' &= r\cos\theta\cos\beta - r\sin\theta\sin\beta \\ &= x\cos\beta - y\sin\beta \\ y' &= r\cos\theta\sin\beta + r\sin\theta\cos\beta \\ &= x\sin\beta + y\cos\beta \end{aligned}$$

answering the question. Let's denote these transformation formulae as follows:

$$\begin{matrix} x \\ y \end{matrix} \rightarrow \begin{matrix} x\cos\beta - y\sin\beta \\ x\sin\beta + \cos\beta \end{matrix}$$

Three-Dimensional Thinking

Think of our rotation as one in three-dimensional space, as a point (x, y, z) rotated about the z-axis through an angle β. In this three-dimensional rotation, the z-coordinate of the point does not change (the "height" of the point is unchanged), but the x- and y-

coordinates transform as a rotation through angle β in their plane. We have

$$\begin{array}{ccc} x & & x\cos\beta - y\sin\beta \\ y & \rightarrow & x\sin\beta + y\cos\beta \\ z & & z \end{array}$$

There is nothing special about a rotation about the z -axis. A rotation about the y -axis must follow analogous formulae.

$$\begin{array}{ccc} x & & x\cos\beta - z\sin\beta \\ y & \rightarrow & y \\ z & & x\sin\beta + z\cos\beta \end{array}$$

And a rotation about the x -axis must be given by

$$\begin{array}{ccc} x & & x \\ y & \rightarrow & y\cos\beta - z\sin\beta \\ z & & y\sin\beta + z\cos\beta. \end{array}$$

At present we have the point

$$\left(R + r\cos\theta,\ 0,\ r\sin\theta\right)$$

on the torus in the xz -plane . A general point on the torus is found by rotating this point about the z -axis through some angle ϕ . This gives a point with coordinates

$$x = \left(R + r\cos\theta\right)\cos\phi$$
$$y = \left(R + r\cos\theta\right)\sin\phi$$
$$z = r\sin\theta.$$

We have found the *parametric equations* of a torus. Every point on the figure is determined by two parameters: the angle θ at which it sits in the "tube" of the torus and the angle ϕ about the axis of the torus (the one through the torus hole) at which it the point sits along the tube.

Using the relation $\cos^2\phi + \sin^2\phi = 1$ we see that $x^2 + y^2 = \left(R + r\cos\theta\right)^2$. Since $R > r$, $R + r\cos\theta$ is always positive and so we can write $\sqrt{x^2 + y^2} = R + r\cos\theta$. Using $\cos^2\theta + \sin^2\theta = 1$ we now deduce

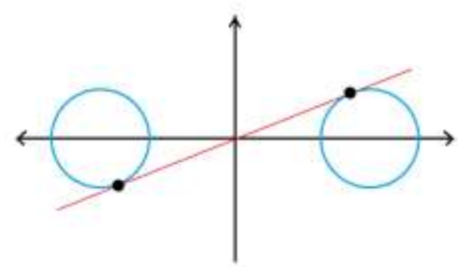
$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2.$$

This is the explicit equation of a torus (that is, is the surface of a perfect bagel). Any point (x, y, z) on the torus satisfies this equation.

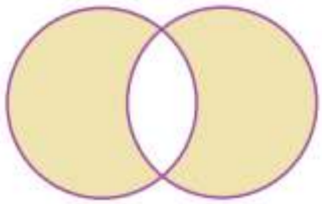
Query: How does one prove the converse? If x , y , and z are three numbers which make this given equation a true number statement, how does one argue that (x, y, z) is a point in the torus?

SLICING AN IDEAL BAGEL: THE PUZZLER

Surprisingly there is indeed a third way to slice an (ideal) bagel to produce a two-perfect circles in the cross-section. It involves cutting at an angle just tangent to two opposite points on the tube of the bagel.



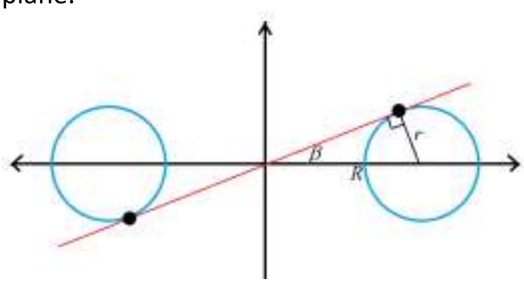
This produces a cross-section of two intersecting circles.



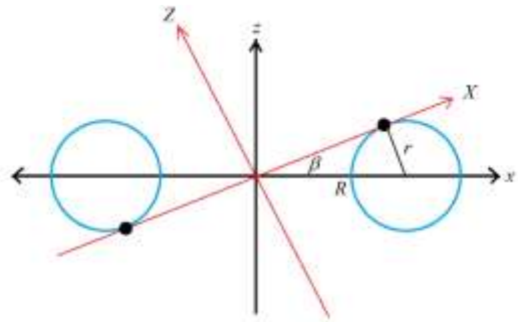
I personally find it very hard to perform such a slice. But we can prove mathematically that a slice made this way does produce two perfect circles.

Warning: Some hefty algebraic work is coming our way!

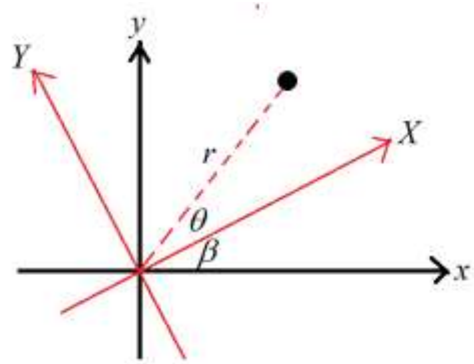
The slicing plane we seek makes an angle $\beta = \arcsin\left(r / R\right)$ with our horizontal xy-plane.



Let's do something wild and change our frame of reference. Let's work with an XYZ-coordinate system offset from our current xyz-coordinate system by a rotation of angle β about the y axis.



An Aside on Rotations of Axes: Suppose we rotate a pair of axes (in two dimensions) about the origin through an angle β. If $\left(X, Y\right)$ are the coordinates of the point in the rotated system, what are its coordinates with respect to the original system?



Again, it is easiest to think in terms of polar coordinates.

If, with respect to the XY-axes, the point has coordinates

$$X = r\cos\theta$$
$$Y = r\sin\theta,$$

then we see from the diagram that the same point has coordinates

$$x = r\cos\left(\theta + \beta\right)$$
$$y = r\sin\left(\theta + \beta\right)$$

with respect to the xy-system. We have been here before! We get from the addition formulae

$$x = X\cos\beta - Y\sin\beta$$
$$y = X\sin\beta + Y\cos\beta.$$

In summary:

If a point (X, Y) in the XY - coordinate system is interpreted as a point (x, y) in the xy -coordinate system, then, from the xy -perspective, it appears as though the point (X, Y) has been rotated β degrees.

$$\begin{matrix} X \\ Y \end{matrix} \rightarrow \begin{matrix} x = X\cos\beta - Y\sin\beta \\ y = X\sin\beta + Y\cos\beta \end{matrix}$$

Suppose (X, Y, Z) is a point on the torus in the new XYZ -coordinate system. Then, as a point in the xyz -system it has coordinates

$$\begin{aligned} x &= X\cos\beta - Z\sin\beta \\ y &= Y \\ z &= X\sin\beta + Z\cos\beta. \end{aligned}$$

We know that the equation

$$\left(\sqrt{x^2+y^2}-R\right)^2 + z^2 = r^2$$

holds for the numbers x, y, and z, and so we must have

$$\left(\sqrt{(X\cos\beta - Z\sin\beta)^2 + Y^2} - R\right)^2 + (X\sin\beta + Z\cos\beta)^2 = r^2$$

This is the equation of the torus in the XYZ -system.

Fortunately, we are interested in the slice that has $Z = 0$ (that is, we are slicing along the XY -plane). So the set of points of the torus that lie on this slicing plane are given by the equation

$$\left(\sqrt{X^2\cos^2\beta + Y^2} - R\right)^2 + X^2\sin^2\beta = r^2.$$

Is this the equation of two intersecting circles? It certainly does not look like it!

Let's follow our noses at this point and see if we can grind through the tedious algebra to make the equations of two circles appear.

It seems appropriate to expand this expression. Doing so gives

$$X^2 + Y^2 - 2R\sqrt{X^2\cos^2\beta + Y^2} + R^2 = r^2.$$

Isolating the square root and squaring seems natural. That gives

$$4R^2\left(X^2\cos^2\beta + Y^2\right) = \left(X^2 + Y^2 + R^2 - r^2\right)^2.$$

Hmm. The $\cos\beta$ term hasn't gone away.

Ah! But looking back at the picture that gave us $\sin\beta = \dfrac{r}{R}$ we deduce that $\cos\beta = \dfrac{\sqrt{R^2 - r^2}}{R}$. Substituting this in gives

$$4\left(X^2\left(R^2 - r^2\right) + R^2Y^2\right) = \left(X^2 + Y^2 + R^2 - r^2\right)^2.$$

Expand all this?

$$\begin{aligned} &X^4 + Y^4 + 2X^2Y^2 - 2R^2X^2 + 2r^2X^2 \\ &- 2R^2Y^2 - 2r^2Y^2 + R^4 - 2r^2R^2 + r^4 = 0 \end{aligned}$$

Okay. Let's pause.

We're expecting to see the equations of two separate circles.

From the symmetry of matters these circles will have centers lying on the X axis. They will be symmetrically placed about the origin. And the two circles will have the same radii.

So we expect then to see two equations of the form

$$X^2+(Y-a)^2=b^2$$

and

$$X^2+(Y+a)^2=b^2$$

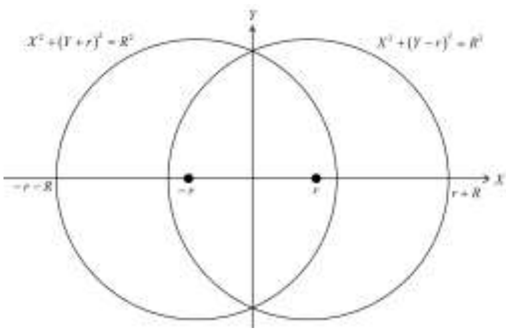
for some values a and b.

Is the mess of an equation we see about equivalent to

$$\left(X^2+(Y-a)^2-b^2\right)\left(X^2+(Y+a)^2-b^2\right)=0$$

for some a and b?

Well … Expand out this dream equation and compare with our mess of an equation. We see that there is a match with $a=r$ and $b=R$. Crazy!



So slicing a bagel tangentially does indeed produce two perfect circles in its cross section (despite my poor abilities in actually slicing a bagel tangentially).

The two intersecting on the torus surface that arise from such a tangential slice are called *Villarceau Circles*. They are named after French mathematician Yvon Villarceau (1813 – 1883).

THE FOUR CIRCLES PROPERTY

Given any point on the torus there are certainly two circles passing through it: the latitudinal and longitudinal ones.

If we rotate a Villarceau circle about the axis of the torus (the one through its hole) we see that it sweeps over every point of the torus. Thus, for any given point on the torus, there are two additional circles passing through it: two Villarceau circles.

Thus every point on a torus has four surface circles passing through it.

RESEARCH CORNER

Is there a point on a torus with five distinct circles passing through it?

Mathematicians have proven that if a (sufficiently nice) surface sitting in three-dimensional space has the property that for any point on the surface there precisely four distinct circles passing through it, then that surface must be a torus. Is there straightforward way to explain this?

Prove that if a (sufficiently nice) surface has the property that for any point on it there are at least five distinct circles passing through it, then the surface must be a sphere (and the number of distinct circles through each point is actually infinite).

Are there some non-sufficiently nice surfaces (fractal-like surfaces, perhaps) which serve as counter examples to all the above claims for surfaces in general?

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