WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2020

The classic game of NIM is an ancient game played with piles of pebbles. Its origin is unknown. But it was only relatively recently in 1901 that the full mathematics behind the game was explored and explained. This was done by American mathematician Charles Bouton, who also coined the name "NIM."

This essay was first written for the PUZZLES EXPLAINED BY EXPLODING DOTS page (temporarily) [here](#). Not to belittle the mathematics typically written to explain a winning strategy for the game, I was struck by how simple the imagery of a $1 \leftarrow 2$ machine makes that explanation. I thought is was worth sharing that content here too.

Today's puzzle, and essay, is to understand the winning way to play NIM.

THE GAME NIM

The game of NIM starts with three piles of counters on the tabletop between two seated players. One pile contains 3 counters, one 5 counters, and the third 7 counters.

3 = 1 ← 2

5 = 1 ← 2

7 = 1 ← 2

↑ ↑ ↑ ↑
ODD ODD ODD ODD
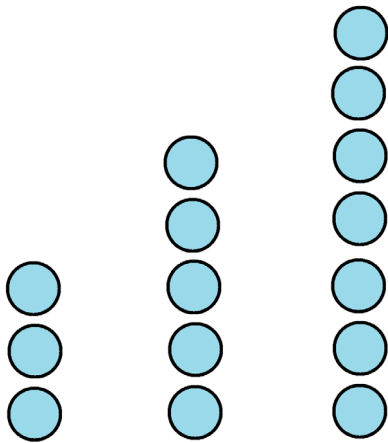
Players take turns removing one or more counters from a single pile. One is permitted to remove *all* the counters from a pile. One is required to take at least one counter.

The person who takes the last counter—thereby leaving the tabletop empty—is the winner.

Try playing a few rounds of NIM with a partner just to get a feel for the game. Switch back and forth for who has the first move.

One can, of course, play this game with any number of piles, each containing any number of counters. To keep the conversation consistent, we'll just discuss this 3-5-7 game of NIM for now.

Now for the puzzler.

THIS MONTHS' PUZZLER:

In 1901, American Mathematician Charles Bouton found a curious winning strategy for playing NIM. We present his approach here in terms of codes in a $1 \leftarrow 2$ exploding dots machine (aka binary codes).

Represent the count of pebbles in each pile in a $1 \leftarrow 2$ machine with the machines stacked on top of one another as shown.

At present, each column in this picture contains an odd number of dots.

A NIM move will change this diagram and change the even/odd-ness of some or of all of the columns. For example, removing two pebbles from the middle pile produces this picture.

3 = 1 ← 2

3 = 1 ← 2

7 = 1 ← 2

↑ ↑ ↑ ↑
ODD EVEN EVEN ODD

a) Find a NIM move from the original 3-5-7 diagram that the first player can take that, instead, gives a new diagram with each column having an even count of dots.

b) From your new diagram with each column possessing an even count of dots,

explain why, in whatever NIM move the second player now takes, she is sure to create a diagram with at least one column containing an odd number of dots.

c) Prove, when presented with a diagram containing at least one column with an odd count of dots, it is always possible to make a single NIM move that gives a diagram with each column containing an even count of dots.

d) Prove, when presented with a diagram with all columns containing an even count of dots, every NIM move is sure to create a diagram with at least one column possessing an odd count of dots.

e) Explain why the first player of the 3-5-7 NIM game can be sure to win the game.

EXPLAINING THE STRATEGY

No matter the number of piles or the size of the piles it is always possible to find a NIM move that turns a diagram with one or more columns containing an odd count of dots into a diagram with all columns containing an even count of dots.
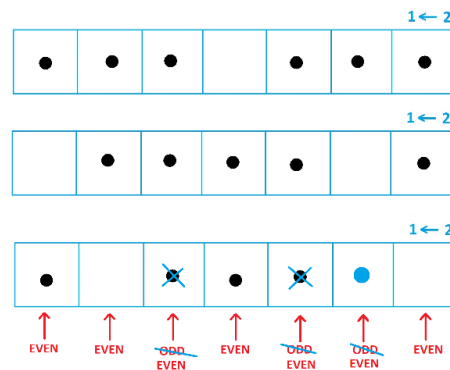
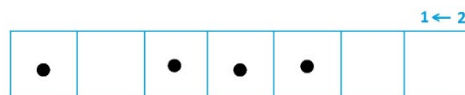Consider this diagram with three piles each containing a large number of pebbles.
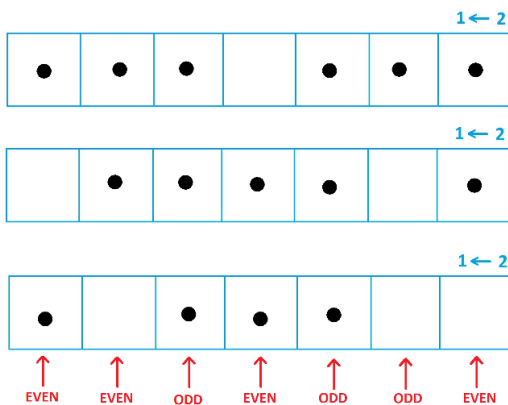
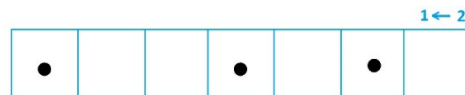Choose any dot in the leftmost "odd column" and delete it.
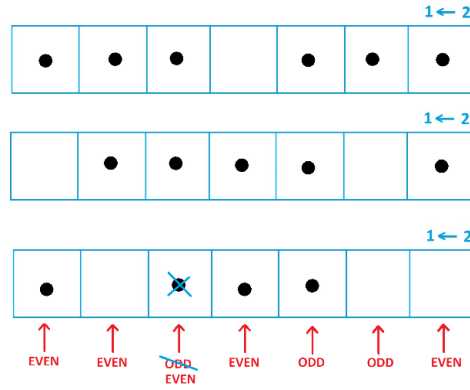
Next, either add or delete dots to the right of the deleted dot to give each column an even count of dots.

In this picture we have change the third pile from

to

that is, we changed the third pile from 92 pebbles to 74 pebbles, a smaller number. This corresponds to taking pebbles out from a pile and so is a valid NIM move.
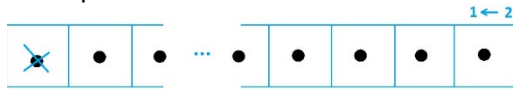
But this begs the question:

In a $1 \leftarrow 2$ machine code, if we delete a dot and change some or all of the boxes to its right, is the result sure to be the code of a smaller number?

What's the worst possible case scenario? It would deleting just the one dot and adding the maximal number of dots possible to its right.
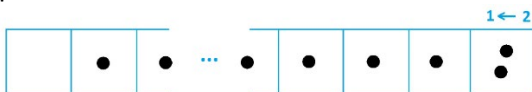
Does changing this picture
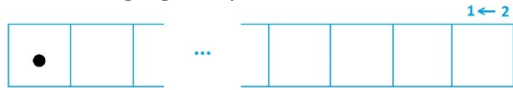
to this picture

give us a smaller number and so still represent the act of removing pebbles from a pile?

Well, yes!

Can you see that with unexplosions that the first of these pictures is equivalent to this picture?

And so it represents a number one larger than the second picture!

We have

Given a NIM game with a matching diagram with some columns containing an odd count of pebbles, there is always a valid NIM move that will yield a matching diagram with all columns containing an even count of pebbles.

Now, on the other hand, suppose you are handed a NIM scenario with matching diagram all of whose columns possess an even count of dots. (Call this an EVEN SCENARIO.) Whatever move you make changes the dots in one row of the diagram. In fact, we can be sure that the state of at least one box will change (if not, you haven't made a move). If a box loses a dot or if a box gains a dot, the column containing that box has turned to an "odd column."

Given a NIM game with a matching diagram with all columns containing an even count of pebbles, any move is sure to produce a diagram with at least one column containing an odd count of pebbles.

Let's call a NIM scenario with at least one column containing an odd count of pebbles an ODD SCENARIO.

You, as a savvy player, now have a winning strategy if presented with an ODD SCENARIO: always play to give your opponent an EVEN SCENARIO. Your opponent will be forced to hand you back an ODD SCENARIO, which means there is at least one pebble remaining on the table. That is, your opponent simply cannot present you an empty set of piles. You have to be the one who does creates that, which means your win is certain with this strategy.

NEXT CHALLENGE: Can you create a mental schema so that you can do all the binary manipulations swiftly in your head while you play?

> **Challenge:** Analyze misère NIM where the object of the game is NOT to win. Might there ever be a strategy for a player to ensure she never picks up the last pebble?

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RESEARCH CORNER

The game of NIM is analyzed through the binary codes of numbers, that is, by codes using just the digits 0 and 1 based on the powers of two.

But there are also codes for numbers, again just using the digits 0 and 1 based on the powers of negative two. These representations are called *negabinary* code.

They work by using a $-1 \leftarrow 2$ machine.

In such a machine, two dots in one box explode away to be replaced by one antidot, one place to their left, and two antidots in one box explode away to be replaced by a dot, one place to their left.
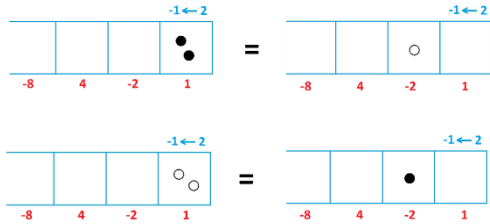

This is machine certainly gives codes for numbers in base negative two, with each box containing at most one dot or one antidot.

But we can go little bit further. Any box that contains an antidot, can be replaced by two single dots. We see this by adding a dot/antidot pair and performing one explosion.
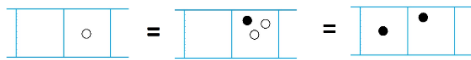
This means that any number, positive or negative, can be represented as a code in a $-1 \leftarrow 2$ machine with nothing but dots with at most one dot per box. Either place N dots or N antidots in the rightmost box (for the positive integer N or the negative integer $-N$), explode away pairs of dots and antidots from the rightmost box. If an antidot is left behind, replace it with two dots as shown above, and then repeat this procedure for the second box from the right, then the third box from the right, and so on. What will be left behind is a representation of N or $-N$ with single dots in boxes, that is, a representation of N or $-N$ as a sum of single powers of -2.
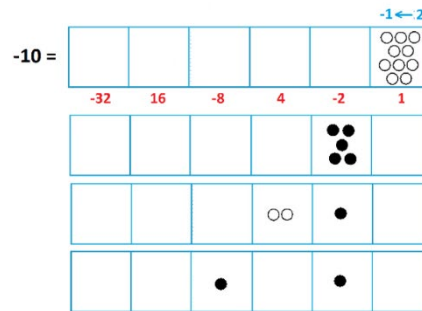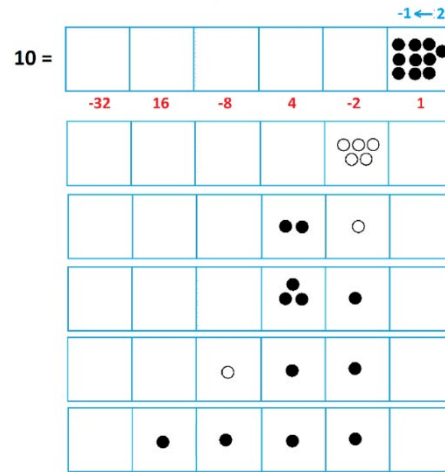
Positive ten is 11110 in base negative two.
Negative ten is 1010 in base negative-two

Challenge: Prove that each integer has a *unique* negabinary representation.

Finally, here's my—very loose—open challenge. Can one invent a NIM-like game, perhaps using two types of pebbles, with a winning strategy based on analysing the negabinary codes of the counts of pebbles in piles?

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