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WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2016

EXCITING NEWS!

There is something big afoot! Check out:

THE GLOBAL MATH PROJECT
Uplifting Mathematics for All

www.theglobalmathproject.org

During the week starting 10.10.2017, we, a team of seven, with a growing team of ambassadors, plan to thrill one million students, teachers, and adults with an engaging piece of mathematics, and have that reach grow significantly each year thereafter! Our goal is to initiate a fundamental paradigm shift in how the world perceives and enjoys mathematics.

We will show how to declutter math content to reveal its meaning, its story, and its joy. And Exploding Dots will be our first rollout experience. (A proven winner!)

Please consider supporting the project – by participating, by subscribing to our newsletter, by offering a service, by spreading the word, by giving a donation, by giving a thumbs up, or some or all of the above. The Global Math Team would be so very grateful if you joined us. (Become an ambassador!) Thank you!

This essay starts with a puzzle I posed on twitter, which led to a next puzzle, and a next. My thanks to @republicofmath and @sted304A for computing very very triangular numbers, and to @daveinstpaul for an elegant idea giving a proof.

THIS MONTH'S PUZZLER:
The sequence of numbers that are the sum of two distinct powers of two begins:

$$3 = 1+2$$
$$5 = 1+4$$
$$6 = 2+4$$
$$9 = 1+8$$
$$10 = 2+8$$
$$12 = 4+8$$

What is the 50th number in this list?

NUMBERS IN BINARY

The very first machine we discuss in the story of Exploding Dots is the $1 \leftarrow 2$ machine. (See lessons 1.1 and 1.2 of http://gdaymath.com/courses/exploding-dots/) There we learn that every counting number can be expressed in a code of 0 s and 1s, which corresponds to writing each number as a sum of distinct powers of two. For instance:

$$13 \;\leftrightarrow\; 1101 \;\leftrightarrow\; 8+4+\cancel{2}+1.$$

We call these representations of number *binary* or *base two* representations.

Question: Is it obvious that each binary representation of a counting number is unique?

We can conduct arithmetic in base two in much the same way as we conduct arithmetic in base ten: we just now "carry the two."

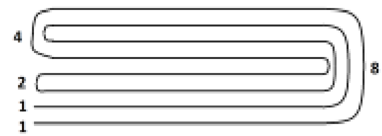
$$\begin{array}{rr} & 1011 \\ + & 110 \\ \hline = & 10001 \end{array} \qquad \begin{array}{rr} & 1111111111 \\ + & 1 \\ \hline = & 10000000000 \end{array}$$

The second example suggests

$$1+1+2+2^2+2^3+\cdots 2^n = 2^{n+1},$$

which is a special case of the geometric series formula. (What's the base-ten version of this observation?)

We can also see this formula from repeatedly folding a strip of paper right to left.



$$1+1+2+4=8$$

The powers of two, $1, 2, 4, 8, \ldots$ are the numbers with precisely one 1 in each of their binary representations.

The opening puzzle asks for the fiftieth number with precisely two 1s in its binary representation. The first $1+2+3+4=10$ of these numbers are

$11 \leftrightarrow 3$	$101 \leftrightarrow 5$	$1001 \leftrightarrow 9$	$10001 \leftrightarrow 17$
	$110 \leftrightarrow 6$	$1010 \leftrightarrow 10$	$10010 \leftrightarrow 18$
		$1100 \leftrightarrow 12$	$10100 \leftrightarrow 20$
			$11000 \leftrightarrow 24.$

In general, there are n such numbers with $n+1$ digits in base two: have a 1 followed by n digits with just one of those digits a 1. There are thus $1+2+3+\cdots+n$ numbers with two 1s in their binary representations with at most $n+1$ digits.

As $1+2+3+\cdots+8+9=45$, the 45th number in the desired list is ten digits long in binary and is

$$11\ 0000\ 0000 \leftrightarrow 512+128=640.$$

The fiftieth such number is eleven digits long in binary and is

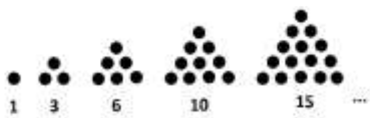
$$100\ 0001\ 0000 \leftrightarrow 1024+16=1040.$$

Challenge: What is the fiftieth number that is the sum of three distinct powers of two?

Challenge: What is the million-and-first number with an odd number of 1s in its binary representation?

VERY TRIANGULAR NUMBERS

The sequence of triangular numbers begins $1,3,6,10,15,\ldots$



and the N th triangular number is

$$1+2+3+\cdots+N=\frac{N(N+1)}{2}.$$

Let's call a triangular number *very triangular* if it has a triangular number of 1s in its binary representation

Very triangular numbers exist: 1, 21, 28, 55, and 190 are five examples. Are there more?

THEOREM: *There are infinitely many very triangular numbers.*

Proof: Look at the 2^n+3-th triangular number for $n\geq 4$. It is

$$\begin{aligned}&\frac{(2^n+3)(2^n+4)}{2}\\ &=(2^n+2+1)(2^{n-1}+2)\\ &=2^{2n-1}+2^{n+1}+2^n+2^{n-1}+2^2+2^1\end{aligned}$$

and has 6 - a triangular number – of 1s in its binary representation.

We can go further.

Each of the triangular numbers $2^{2n-1}+2^{n+1}+2^n+2^{n-1}+2^2+2^1$ has, in binary, $2n$ digits, six of which are 1s leaving $2n-6$ digits as zero.

And there are infinitely many even triangle numbers. Thus we can find infinitely many values $n\geq 4$ with $2n-6$ a triangular number. (For example, $n=6$ gives $2n-6=6$, and $n=8$ gives $2n-6=10$.)

There is a pattern to which values n give $2n-6$ a triangular number. What is the pattern?

This means that there are infinitely many *very very triangular* numbers, that is, infinitely many triangular numbers, which, when written in binary, have a triangular number of 1s AND a triangular number of (non-leading) 0 s.

The smallest very very triangular number is 276, which is 100010100 in binary. Other examples include 378, 435, and 2278 (and this last one comes from placing $n=6$ in our previous formula).

RESEARCH CORNER

As far as I am aware, no one has discussed the notion of very triangular and very very triangular numbers before. We're now in (very) original research territory!

The sequence of very triangular numbers begins $1, 21, 28, 55, 190,\ldots$. Is there a formula for the N th very triangular number? (The formula in this essay skips over most of these numbers.)

The sequence of very very triangular numbers begins

$$1,276,378,435,2278,2701,\ldots.$$

(Should we include 1?) Is there a formula for the N th very very triangular number?

Explore very square and very very square numbers.

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