CURRICULUM INSPIRATIONS: www.maa.org/ci
Math for America DC: www.mathforamerica.org/DC

Innovative Online Courses: www.gdaymath.com
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Wild COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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A number of folk across the globe are developing materials for their students based on the ideas developed in EXPLODING DOTS and PERMUTATIONS & COMBINATIONS and QUADRATICS at www.gdaymath.com. (Only these three courses are up and running so far, but many more are coming – along with some streamlining of these courses!)

A next goal is to create on this site a repository of resources folk have developed and would like to share.

If you have worksheets, assignments, lesson plans or ideas you would like to contribute, email me at tanton.math@gmail.com and I will make sure they appear once all is in place.

PUZZLER: We have:

$$8^2 = 64$$

$$12^2 = 144$$

Is there a square number that ends with three consecutive 4 s? Four consecutive 4 s?

REPUNITS

A <u>repunit</u> is an integer all of whose digits are 1 (when written in base ten). The first repunit is 1, the second is 11, and the twenty-third is

$$11111111111111111111111.$$

Let $1(N)$ denote the N th repunit.

The term "repunit" was coined by Albert Bailer in 1966 in his studies of the decimal expansions of fractions. Bailer observed, for instance, that for each prime p greater than 5 the period of the decimal expansion of $\frac{1}{p}$ is the length of the smallest repunit that is a multiple of p.

For example, $\frac{1}{7} = 0.\overline{142857}$ has period six and $1(6) = 111111 = 7 \times 15873$ is the first repunit divisible by 7.

> **Do-able Challenge:** Prove that for each prime p different from 2 and 5, there is a repunit divisible by p.
>
> <u>Hint</u>: Why must there be two repunits that leave the same remainder upon division by p? Why must their difference be a multiple of p? What does their difference look like? Why is excluding 2 and 5 from our considerations important?

Only five repunits are known to be prime: $1(2)$, $1(19)$, $1(23)$, $1(317)$, and $1(1031)$. No one on this planet knows if the list of prime repunits is finite or infinite.

> **Challenge:** Earn world fame by settling this open question!

It is straightforward to see that if N is a multiple of a, then $1(N)$ is a multiple of $1(a)$:

$$1(N) = \overbrace{111..11}^{a}\ \overbrace{111..11}^{a} \cdots \overbrace{111..11}^{a}$$
$$= 1(a) \times \overbrace{000..01}^{a}\ \overbrace{000..01}^{a} \cdots \overbrace{000..01}^{a}$$

For example,

$$1(6) = 111\ \ 111\ = 111 \times 1001$$
$$= 11\ \ 11\ \ 11\ = 11 \times 10101$$

So to hunt for prime repunits, we need only look among the prime indexes.

If N is not prime, then $1(N)$ won't be prime.

> **Do-able Challenge:** Prove that there is only one prime repunit in base four, namely 11_4 (which is the number $4+1=5$).

> **Challenge:** The number 31 is a repunit in two different bases: in base five it is 111_5 and in base two it is 11111_2. The number 8191 is a repunit in base ninety and in base two. These are the only two known examples of integers (different from 1) that are repunits in two different bases. For more world fame, prove the *Goormaghtigh Conjecture,* which states that these are the only two non-trivial integers with this property (or show the conjecture to be wrong!)

Let's call a repunit $1(N)$ *special* if it is divisible by N. For example:

$1(3) = 111 = 3 \times 37$ is divisible by 3 and so is special.

$1(9) = 111111111 = 9 \times 12345679$ is divisible by 9 and so is also special.

> **Do-able Challenge:** Prove that if $1(N)$ is special, then so is $1(3N)$.

So from $1(1)=1$ being special we get that $1(3)$, $1(9)$, $1(27)$, $1(81)$, and so on, are special as well. But these are not the only special repunits: $1(111)$ is also special, for instance.

If $1(N)$ is special, then so is $1(1(N))$.

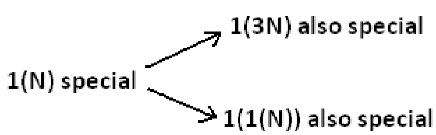
Reason: If $1(N)=kN$, then $1(1(N))=1(kN)$ looks like:

$$1(kN) = \overbrace{111..11}^{N}\,\overbrace{111..11}^{N}\cdots\overbrace{111..11}^{N} \quad \text{k copies}$$

$$= 1(N) \times \overbrace{000..01}\,\overbrace{000..01}\cdots\overbrace{000..01}$$

We see that $1(1(N))$ is a multiple of $1(N)$ and so is also special.

So we now have two ways to generate special repunits from known ones:



Challenge: Starting with $1(1)$ being special, do these two methods of generation generate <u>all</u> the special repunits?

Challenge: I recently tweeted the following puzzle: *Is $1(N)$ ever divisible by N^2 (for $N>1$)?* I admit that I do not know the answer.

My esteemed colleague @republicofmath conducted a computer search and found this to never be the case for $N=2,\ldots,10000$.
How about for $N>10,000$?

Comment: My colleague also showed that $1(487)$ is a near miss: it leaves a remainder of only 1 upon division by 487^2.

SQUARE REPUNITS AND SQUARE REPDIGITS

We have:

$1(1)=1$ *is the only repunit that is a perfect square.*

Reason: If $111...11=n^2$, then n must be an integer that ends with either 1 or 9. Thus $n=10k\pm 1$ for some integer k. Then $n^2=100k^2\pm 20k+1$, which shows that n^2 is an integer with second-to-last digit even. The only repunit with an even second-to-last digit is $1(1)$.

Challenge: Is $1(1)$ the only repunit that is a perfect cube?

Comment: I once thought I had a proof that this was the case, but then I found an error in my reasoning. (Drats!) Is the answer to this question even clear from the literature?

Let's generalize matters a little. A <u>repdigit</u> is an integer whose base ten representation consists of a single digit. Let's use the notation $4(N)$ to denote the N-digit number composed solely of 4s, $2(N)$ the N-digit number composed solely of 2s, and so on.

Do-able Challenge: Prove that 1, 4, and 9 are the only repdigits that are perfect squares.

Comment: John McCarthy discovered that repdigit squares do exist in other bases. For example, $1111_7=20^2$ and $777_{18}=49^2$.

Another Comment: $6(N)$ is divisible by N^2 for $N=3$ and 111. $9(N)$ is divisible by N^2 for $N=3,9,111,333$. (Thanks again to@republicofmath for these!)

SQUARES WITH REP-ENDINGS

A square number can certainly end with one, two, or three consecutive 4s. For example:

$2^2 = 4$, $12^2 = 144$, and $38^2 = 1444$.

And this is the best we can hope for!

THEOREM: *No square number ends with 2, 3, 7, or 8. A square number can end with at most one 1, at most one 5, at most one 6, at most one 9, and at most three consecutive 4s. A square number can end with any even number of 0s.*

N	Max Consecutive Ending Digits
1	1
2	0
3	0
4	3
5	1
6	1
7	0
8	0
9	1
0	any even count

Reason: Write

$$N = 1000n + 100a + 10b + c$$

where a, b and c are single digits, and n is an integer. (Thus the last three digits of N are abc.)
Then

$$N^2 = \textit{ten thousands} + 100b^2 + c^2 + 2000nc + 200ac + 20bc + 2000ab$$
$$= \textit{ten thousands} + 1000(2nc + 2ab) + 100(b^2 + 2ac) + 10(2bc) + c^2$$

Thus the final four "digits" of N^2 are:

$$2nc + 2ab \mid b^2 + 2ac \mid 2bc \mid c^2$$

but there might be carrys involved.

For N^2 to end with a 1 we need $c = 1$ or $c = 9$. If $c = 1$, the second-to-last digit is $2b$, which is even, and cannot end with another 1. If $c = 9$, then the second-to-last digit is $18b + 8$ (there's a carry), which again is even and cannot end with a 1. Thus a square number cannot end with two 1s.

There is no single digit c such that c^2 ends with a 2, and so N^2 can never end with 2. Also, as no single digit c has c^2 ending with 3, 7, or 8, N^2 will never end with these digits either.

For N^2 to end with a 5, we need $c = 5$. In this case its second-to-last digit will be $10b + 2$, even. N^2 cannot end with two 5s.

For N^2 to end with a 6, we need $c = 4$ or $c = 6$. In these cases the second-to-last digit will be either $8b + 1$ or $12b + 3$, each odd. N^2 cannot end with two 6s.

For N^2 to end with a 9, we need $c = 3$ or $c = 7$. In the first case its second-to-last digit will be $6b$, even. N^2 cannot end with two 3s. In the second case, the second-to-last digit will be $14b + 4$, even. N^2 cannot end with two 9s.

We are now left to prove that no square number ends with four 4s. Let's go through this case-by-case.

Looking at the final digit of N^2 there are two possible values for c:

<u>$c = 2$</u>: The final digits of N^2 are

$$4n + 2ab \mid b^2 + 4a \mid 4b \mid 4.$$

<u>$c = 8$</u>: The final digits of N^2 are

$$16n + 2ab \mid b^2 + 16a \mid 16b + 6 \mid 4.$$

For the second-to-last digits to also be 4 s we have the cases:

$\underline{c=2}, \underline{b=1}$: The final digits of N^2 are

$$4n+2a \mid 1+4a \mid 4 \mid 4.$$

$\underline{c=2}, \underline{b=6}$: The final digits of N^2 are

$$4n+12a \mid 38+4a \mid 4 \mid 4.$$

$\underline{c=8}, \underline{b=3}$: The final digits of N^2 are

$$16n+6a \mid 14+16a \mid 4 \mid 4.$$

$\underline{c=8}, \underline{b=8}$: The final digits of N^2 are

$$16n+16a \mid 77+16a \mid 4 \mid 4.$$

The first and fourth cases listed here have odd third-to-last entries and so these entries cannot end with a 4.

In the second case with $c=2$, $b=6$, there are two possible values for a that give a third-to-last digit of 4: $a=4$, $a=9$. And in the third case with $c=8$, $b=3$, we have $a=0$ or $a=5$.

These give:

$\underline{c=2}, \underline{b=6}, \underline{a=4}$:

The final digits of N^2 are

$$53+4n \mid 4 \mid 4 \mid 4.$$

$\underline{c=2}, \underline{b=6}, \underline{a=9}$:

The final digits of N^2 are

$$125+4n \mid 4 \mid 4 \mid 4.$$

$\underline{c=8}, \underline{b=3}, \underline{a=0}$:

The final digits of N^2 are

$$1+16n \mid 4 \mid 4 \mid 4.$$

$\underline{c=8}, \underline{b=3}, \underline{a=5}$:

The final digits of N^2 are

$$39+16n \mid 4 \mid 4 \mid 4.$$

In all cases the fourth-to-last entries are odd and so can never end with a 4. Thus a square number can never end with four 4 s.

Do-able Challenge: Why can't a square number end with an odd count of zeros?

RESEARCH CORNER: Prove that a cube number can end with at most one 2, with at most two 4 s, with at most one 5, or with at most one 6.

Can a cube number end with any number of consecutive 1s? Any number of consecutive 3 s, 7 s, 8 s, or 9 s?

N	Max Consecutive Ending Digits
1	no bound?
2	1
3	no bound?
4	2
5	1
6	1
7	no bound?
8	no bound?
9	no bound?
0	no bound!

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