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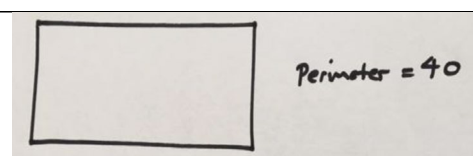
CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2020

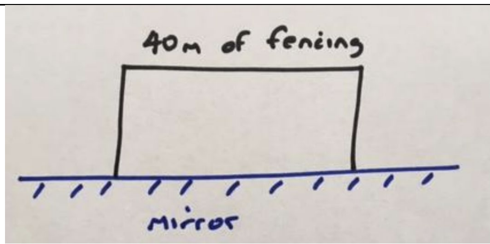
THIS MONTHS' PUZZLER:

Here are three classic school-book problems. How might you solve them in one unified way?

Classic Example 1: *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. What should the dimensions of the rectangle be to obtain a pen of maximal possible area?*

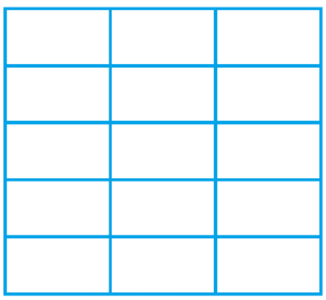


Classic Example 2: *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.*



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Classic Example 3: *Another farmer has 40 meters of fencing and wants to use it all to make a rectangular pen divided into* 15 *small pens as shown.*

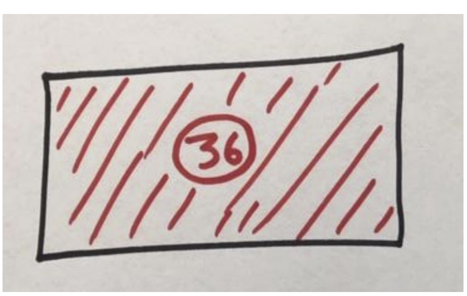


What should the dimensions of the large rectangle be in order to obtain a pen of overall maximal area?

THE POWER OF SYMMETRY

This section of material is closely based on notes from an upcoming College Algebra course.

Here's a rectangle. I tell you its area is 36 square units. What can you tell me about the rectangle?



Well… nothing! It might be a 4-by-9 rectangle, or a 2-by-18 rectangle, or have dimensions 4½-by-8. You don't know. You only know that its area is 36 square units.

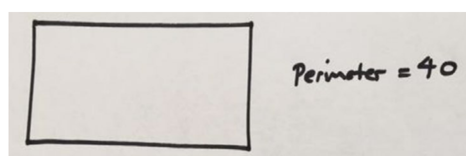
But suppose I now add one more piece of information about this rectangle, that it is *symmetrical*. Then suddenly all uncertainties vanish and you know everything about the rectangle, namely, that it is a 6-by-6 square.

This simple example illustrates the power of symmetry in mathematics. Often the introduction of symmetry in a scenario collapses unknown information into crystalline precise information.

Principle: Mathematicians recognize the power of symmetry. Symmetry is a mathematician's friend!

This essay outlines the power of utilizing symmetry in solving optimization problems. We'll do this by presenting and solving a number of illustrative examples, starting with the three of the opening puzzler.

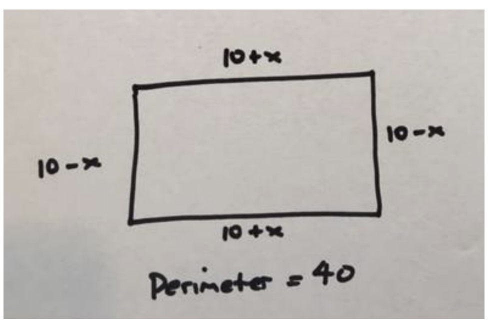
Example 1: *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. What should the dimensions of the rectangle be to obtain a pen of maximal possible area?*



Answer: The usual approach is to label the two side lengths of the pen-to-be with two variables, say x and y. Then we have $2x+2y=40$ and we 're being asked to maximize the value of the area $A=xy$ under this constraint. Noticing that $y=20-x$, we can write $A=x(20-x)=-x^2+20x$. This quadratic has maximal value at $x = 10$ (using the theory of quadratics from algebra class, or using calculus) and for this value, $y = 10$ as well. Thus the pen of maximal area is the ten-by-ten square, the symmetrical situation.

But suppose we considered the symmetrical situation first. After all, if the pen of maximal area is not a square, then its ninety-degree rotation would be a second rectangle of maximal area, and it just doesn't feel right that there would be two solutions to a problem like this. (Just a hunch!)

Let's label the sides of a general pen by how much it deviates from being square. One of its sides will be longer than 10 meters and the other shorter than 10 meters and so let's consider a $10+x$-by-$10-x$ pen for some number x.



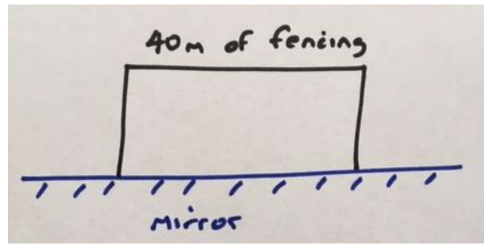
The area this pen is

$$(10+x)(10-x) = 100 + 10x - 10x - x^2$$
$$= 100 - x^2.$$

And clearly $100 - x^2$ is maximal in value for $x = 0$, the pen that doesn't deviate at all from being square!

Basing one's work off of symmetry often leads to less work.

Example 2: *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.*



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Principle: Mathematicians will attempt to push the results of a previously solved problem to help with a new problem.

Answer 1: Geometric Symmetry

This problem is not-symmetrical as the pen has two vertical sides and only one horizontal side. The problem we discussed in Example 1 was symmetrical with two sides of each type.

But if one looks into the mirrored wall it will look as though we have a rectangular pen, double the area and double the perimeter (80 meters), using four full sides of fencing.

We know the answer to the symmetrical problem is a symmetrical square. So pen with mirrored reflection must be a 20 meter-by-20 meter square to obtain maximal area. But the actual pen without the reflection is half of this, a 10 meter-by-20 meter pen.

She should build a 10-by-20 pen, half a square.

Answer 2: Algebraic Symmetry

The question and solution of Example 1 can be summarized:

Given a fixed value for a sum $x+y$, the product xy is maximized when $x=y$.

Each of the three mathematical expressions in this summary is symmetrical. (Switching x and y leaves them unchanged.)

For Example 2, if we label the horizontal length of the pen-to-be as x and each of the two vertical lengths y, then we are being asked

Given a fixed value for the sum $x+2y$, find when the product xy is maximized.

We don't have symmetrical expressions throughout. Can we make it so?

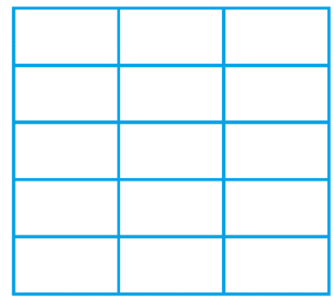
Principle: **Mathematicians engage in "wishful thinking." If she wishes a certain property or scenario to hold, she will work to make it so!**

What if we regard x and "$2y$" as the two variables in this problem? After all, the product xy will be at a maximum precisely when $2xy$, that is, when $x(2y)$ is at a maximum. We know

Given a fixed value for a sum $x+(2y)$, the product $x(2y)$ is maximized when $x=2y$.

Since $x+2y=40$, then gives $x=20$ and $2y=20$, that is, $y=10$, matching our previous solution.

Example 3: *Another farmer has 40 meters of fencing and wants to use it all to make a rectangular pen divided into 15 small pens as shown.*



What should the dimensions of the large rectangle be in order to obtain a pen of maximal area?

Answer: If we label the full length of each of the four vertical lengths of fencing x and each of the six horizontal lengths y, then $4x+6y=40$ and we must maximize xy, or equivalently $24x=(4x)(6x)$. The maximum occurs when $4x=6y$. Thus we must have $4x=20$, giving $x=5$ meters, and $6y=20$, giving $y=3\frac{1}{3}$ meters.

We observe the general idea here.

Suppose a and b are positive values.

Given a fixed value for a sum $ax+by$, the product xy is maximized when $ax=by$.

(Maximizing the product xy is equivalent to maximizing the product $(ax)(by)$.)

If $ax+by=S$, then we specifically have $ax=\frac{S}{2}$ and $by=\frac{S}{2}$ in this maximal scenario.

Practice 4: Of all the points (x, y) on the unit circle $x^2 + y^2 = 1$, which has (have) the largest product xy of coordinates?

Practice 5: Of all the pairs of real numbers that differ by 10, which have the smallest product?

Example 6: Show that for any two positive real numbers a and b we have

$$\frac{a+b}{2} \geq \sqrt{ab}$$

with the inequality being strict when a and b are distinct numbers. (If $a = b$, we clearly get equality.)

Answer: This inequality can certainly be studied solely via algebra. It holds only if $(a+b)^2 \geq 4(ab)$ holds, and this is equivalent to showing that $(a-b)^2 \geq 0$ holds, which It clearly does! (And we have equality only if $a = b$.)

But the inequality also holds from our previous thinking.

Given values a and b let S be their sum: $a + b = S$. Of all the pairs of values with this sum, we know that $\frac{S}{2}$ and $\frac{S}{2}$ give the largest product. Thus $ab \leq \frac{S}{2} \cdot \frac{S}{2}$. That is, $ab \leq \frac{(a+b)^2}{4}$, or, equivalently, $\sqrt{ab} \leq \frac{a+b}{2}$.

The product has this maximal value only if $a = b\left(= \frac{S}{2}\right)$.

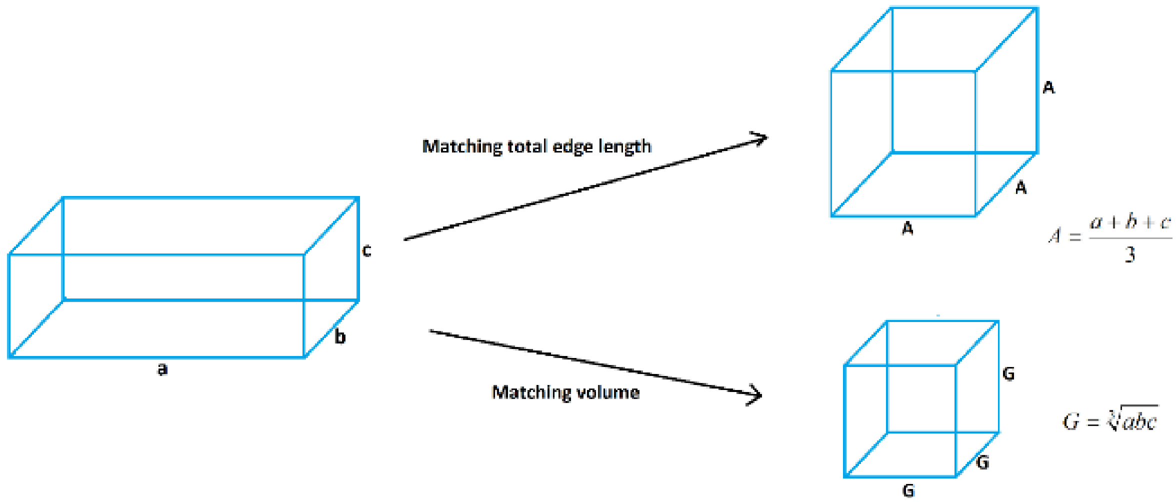
Practice 7: A rectangle has dimensions a meters by b meters.

a) What is the side length of a square with the same perimeter as the rectangle?
b) What is the side length of a square with the same area as the rectangle?
c) Which of these two squares is larger?

ARITHMETIC AND GEOMETRIC MEANS

Consider the wire-frame of a rectangular box of dimensions a meters by b meters by c meters. This box uses $4a + 4b + 4c$ meters of wire in its construction and contains a volume of abc cubic meters.

The cube using the same amount of wire in its frame has edge length $A = \frac{a+b+c}{3}$ (twelve units of this length give $4a + 4b + 4c$) and the cube of the same volume has side length $G = \sqrt[3]{abc}$. These two quantities are, respectively, known as the *arithmetic mean* and the *geometric mean* of the numbers a, b, and c. In general, for any set of positive numbers a_1, a_2, …., a_N their arithmetic mean is set as $A = \frac{a_1 + a_2 + \cdots + a_N}{N}$ and their geometric mean as $G = \sqrt[N]{a_1 \cdot a_2 \cdot \cdots \cdot a_N}$.



The following result is well known and is often used by folk designing problems for mathematics competitions.

THE ARITMETIC-MEAN GEOMETRIC-MEAN (AM-GM) INEQUALITY: *For any set of positive numbers* a_1, a_2, …., a_N, *we have* $A \geq G$ *with the inequality strict if not all the numbers are equal in value.* (If $a_1 = a_2 = \cdots = a_N$, then we see $A = G$.)

Proving the theorem is actually not easy! The proof I like most using very simple tools is due to Y. Uchida and can be found here. But we can look at some low-dimensional cases independently.

We already proved the theorem for $N = 2$ in Example 4.

The case for $N = 4$ follows readily as follows. Observe, for a, b, c, and d positive numbers we have

$$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \frac{\sqrt{ab} + \sqrt{cd}}{2}$$

with equality only if $a = b$ and $c = d$.

Also,

$$\frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{\sqrt{ab} \cdot \sqrt{cd}} = \sqrt[4]{abcd}$$

with equality only if $ab = cd$. Stringing these together gives

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

with equality only if $a = b = c = d$.

Practice 8: Prove the AM-GM inequality for the case $N = 8$.

The case for $N = 3$ takes a different approach.

Let x, y, and z be positive numbers.

From $(x-y)^2 \geq 0$ we get $x^2 + y^2 \geq 2xy$. Similarly, $y^2 + z^2 \geq 2yz$ and $x^2 + z^2 \geq 2xz$. (Equality holds throughout only if $x = y = z$.) Adding these three inequalities gives

$$x^2 + y^2 + z^2 \geq xy + yz + xz.$$

Multiply each side of this inequality by positive number $x + y + z$. After canceling matching terms one sees that it gives

$$x^3 + y^3 + z^3 \geq 3xyz .$$

Now if $x = \sqrt[3]{a}$, $y = \sqrt[3]{b}$, and $z = \sqrt[3]{c}$ for positive numbers a, b, c, then this actually reads

$$a + b + c \geq 3\sqrt[3]{abc} ,$$

which is the AM-GM inequality.

Let's bring this $N = 3$ case back to the setting of wire-frame rectangular boxes. An a-by-b-by-c rectangular box has total edge length $4a + 4b + 4c = 12A$ and a cube of the same volume has total edge-length $12G$. Since $A \geq G$, we have the following geometric interpretation of the AM-GM Inequality.

Of all wire-frame boxes of a given volume, the cube has least total edge-length.

Again, the symmetrical scenario is the optimal scenario.

Example 9: Three positive numbers sum to 1. What is the largest possible value for their product?

Answer: Call the three numbers a, b, and c. We have a symmetrical constraint—$a + b + c = 1$—and seek to optimize a symmetrical expression: abc. We expect the symmetrical scenario, $a = b = c = \frac{1}{3}$ to be the one that gives a maximal product.

The AM-GM inequality confirms this. We have

$$1 = a + b + c \geq 3\sqrt[3]{abc}$$

showing that $abc \leq \left(\frac{1}{3}\right)^3 = \frac{1}{27}$ with this maximum actually being attained for $a = b = c = \frac{1}{3}$.

Example 10: Which two positive numbers a and b with a sum of 12 give the largest value for a^2b?

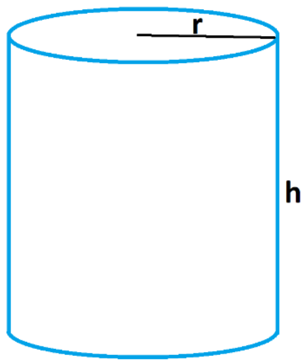
Answer: There is a lack of symmetry in this problem: we have the symmetrical constraint $a + b = 12$, but a non-symmetrical expression a^2b to optimize. It is also not clear how to make use of the AM-GM inequality here: we have, $12 = a + b \geq 2\sqrt{ab}$, but that tells us about the maximal value of ab, not of a^2b.

That a^2b is a product of three terms suggests we might want to consider rewriting the constraint as a sum of three terms.

Principle: Mathematicians engage in "wishful thinking." If she wishes a certain property or scenario to hold, she will work to make it so!

Let's rewrite our constant as $\frac{a}{2} + \frac{a}{2} + b = 12$. Then the AM-GM inequality gives $12 \geq 3\sqrt[3]{\frac{a^2b}{4}}$ showing that $a^2b \leq 256$ with equality actually occurring for $\frac{a}{2} = \frac{a}{2} = b$. This happens for $b = 4$ and $\frac{a}{2} = 4$, that is, $b = 8$.

Example 11: Of all the cylindrical cans of volume 500 cubic centimeters, which uses the least amount of sheet metal to produce? (That is, what are the dimensions of the tin can of this fixed volume with least surface area?)



Answer: Let r be the radius of the can and h its height, in centimeters. We have $\pi r^2 h = 500$. The surface area of the can is $2 \times \pi r^2 + 2\pi r \times h$.

Can we bring the AM-GM inequality into play?

We certainly have

$$2\pi r^2 + 2\pi r h \geq 2\sqrt{4\pi^2 r^3 h}$$

but that is not helpful as we need the term $r^2 h$ to appear, not $r^3 h$.

Let's try looking at the expression for the surface area as $\pi r h + \pi r h + 2\pi r^2$. Then

$$\begin{aligned}\pi r h + \pi r h + 2\pi r^2 &\geq 3\sqrt[3]{2\pi^3 r^4 h^2} \\ &= 3\sqrt[3]{2\pi^3 (500)^2} \\ &= 3\pi (500000)^{\frac{1}{3}}\end{aligned}$$

The surface area has a minimal value of $3\pi (500000)^{\frac{1}{3}}$ which occurs when $\pi r h = \pi r h = 2\pi r^2$. That is, when $h = 2r$.

RESEARCH CORNER

What other classic text-book optimization problems from algebra class or from a calculus course can be solved simply by thinking about symmetry and perhaps by also applying the AM-GM inequality?

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