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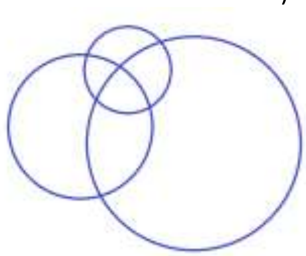
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

March 2017

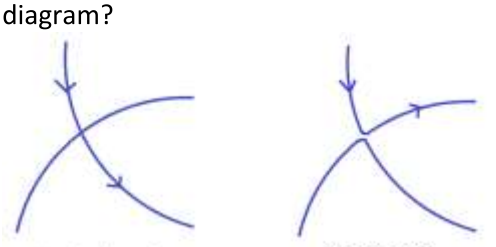
THIS MONTH'S PUZZLER

Draw a set of intersecting circles on a page, making a single connected diagram. (That is, avoid "isolated" circles or isolated collections to circles.)



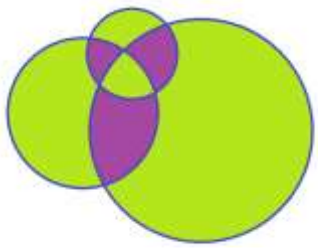
1. Can you redraw your diagram without lifting your pencil from the page (and without going the same edge twice)?
If so, is this the case for all such circle diagrams?

2. Can you redraw your diagram without lifting your pencil from the page AND without crossing through a previously traced line? That is, can you "bounce off" of all intersection points rather than pass through them and still retrace the whole diagram?



If so, is this the case for all such circle diagrams? (This extra idea of retracing diagrams without crossing through intersection points was suggested to me by Colin Wright.)

3. The famous four-color theorem states that the regions of any map drawn on a page can be colored with at most four colors so that any two regions sharing a positive length of boundary are assigned different colors. My picture of intersecting circles is actually colorable with just two colors. Is yours?



Are all such maps composed of intersecting circles two-colorable?

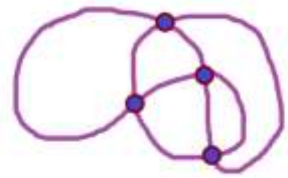
4. Two intersecting circles drawn on a page give 3 bounded regions. Three circles drawn on the page can give at most 7 bounded regions. Find a formula for the largest number of bounded regions N circles drawn on a page could produce. What is the least number they could produce (assuming the diagram drawn is connected)?

5. The diagram of intersecting circles I drew has 6 intersection points and 7 bounded regions. Develop a general theory about the number I of intersection points and the number R of bounded regions for any diagram of intersecting circles.

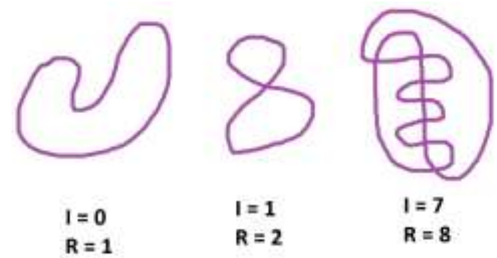
Comment: There may be a number of issues to contend with here. Do you want to allow two circles to touch just at a point of tangency? Do you want to allow multiple circles to pass through the same intersection point?

LOOPS ON A PAGE

Draw a self-intersecting loop on the page.

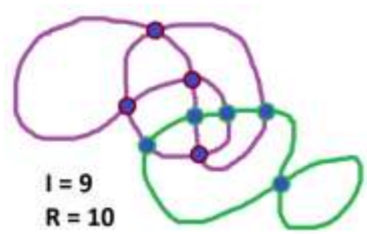


The loop I've drawn has $I = 4$ intersection points and $R = 5$ bounded regions. In fact, all loops one draws on the page are sure to have one more bounded region than intersection points, as long as the loop avoids passing through the same intersection point multiple times and avoids retracing a length of itself.



To see why this is the case, imagine drawing the loop afresh. When your pen crosses a first intersection point, you close off a region of paper and create a first bounded region. Thereafter, each intersection point encountered either splits a previously created region in two or produces a new region. Either way, the count of regions produced increases by 1 with each intersection point encountered. And finally, when your pen returns to start, you close off one final additional bounded region. Thus $R = I + 1$.

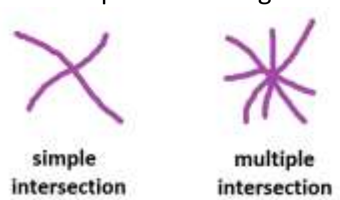
Add a second loop to the picture. Again, if we avoid multiple intersections, the counts of bounded regions and intersection points must still satisfy $R = I + 1$.



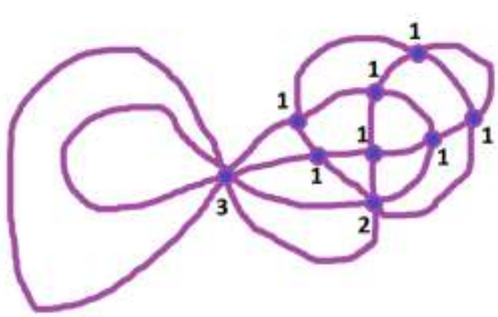
To see this, imagine drawing the second loop afresh in a different color, say green, with the first curve being purple. The first intersection point one encounters with the purple curve fails to create a new bounded region, but all encounters with intersection points thereafter with the purple curve, or with the green curve, do increase the count of bounded regions by 1. But then we create one final bounded region when we close up the green curve. Thus the number of new intersection points created matches the number increase in count of regions produced, and the formula $R = I + 1$ remains balanced and true.

And if we keep adding more intersecting curves to the picture, the formula $R = I + 1$ continues to hold (again assuming we are avoiding multiple use of intersection points).

Multiple Intersections
At an ordinary, simple intersection point one drawn line crosses over another. At a multiple intersection point, multiple lines are drawn on top of a first edge.



Let's give each intersection point a value that matches the number of lines drawn on top of a first line and set I to be the sum of all intersection point values. (And if each intersection point has value 1, then I is just the count of intersection points as before.)



Our previous argument establishes that $R = I + 1$, even for loops with multiple intersection points. (Do you see this?)

Points of Tangency
Points of tangency are indistinguishable from intersection points. If we regard all such points as part of the count for I, then our formula still holds.

> **Question:** Is there a way to adapt our thinking to make a version "$R = I + 1$" to hold even for loops that retrace positive lengths of themselves?
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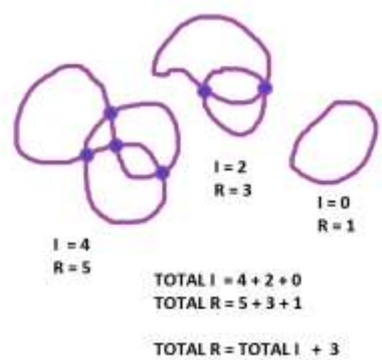
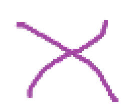
Multiple Components
If a diagram is composed of disjoint pieces, then the formula $R = I + 1$ holds for each component. If there are C components, then adding all C formulas gives $R = I + C$.

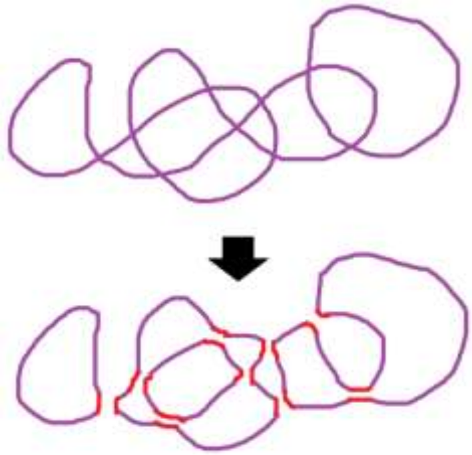


ELIMINATING INTERSECTIONS
Imagine a connected set of self-intersecting loops drawn on the page. Just to keep the discussion straightforward for now, let's assume each intersection point is simple—just one line crossing over a previously drawn edge—and that there are no points of tangency. Thus each intersection point looks like this.

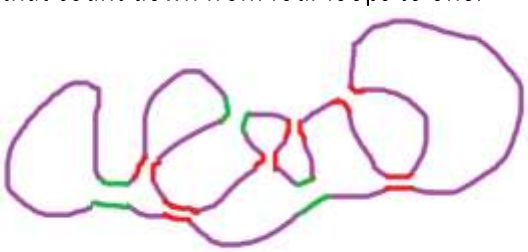


Now replace each such intersection point with one of the following, randomly chosen.

This breaks a given diagram into disconnected, non-intersecting loops.

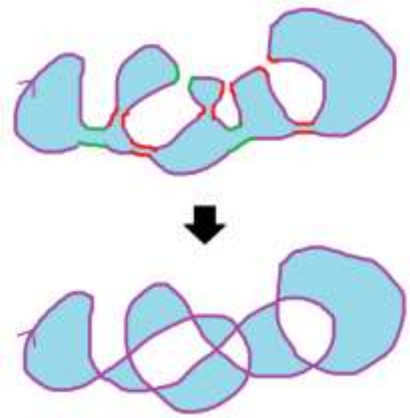


For my diagram, I happened to get four disconnected loops. Switching the choice made at any given intersection point either increases or decreases the count of disjoint loops by one. (Can you see this? Either we break an existing loop into two loops or we combine two previously disjoint loops into one.) So let's go through all the former intersection points and change some choices to decrease the total number of disjoint loops. For my diagram, I can bring that count down from four loops to one.



Actually, every such diagram can be "brought down" to just one loop. This is because a disjoint loop must pass through a former intersection point with another disjoint loop (the original diagram was connected) and changing the choice at that former intersection point now connects those two loops. Thus we can always reduce the count of disjoint loops by one until there is just one loop.

This one loop now shows two wonderful things: how to retrace the original diagram without lifting your pencil from the page and without ever crossing through an intersection point, and how to color the original diagram with just two colors: choose one color for the inside of our newly constructed single loop and one for the outside.

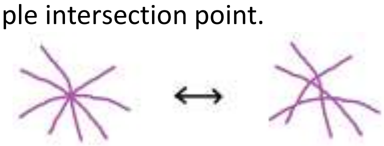


Full Generality

Draw a diagram on the page composed of several intersecting self-intersecting loops. Allow multiple intersection points and points of tangency (but avoid retracing positive lengths of previously drawn curves). If your diagram has C components and I intersection points (counted with multiplicity), then we have seen that the total number of bounded region R on the page is $R = I + C$.

If $C > 1$, that is, if your diagram has more than two components, then it is clearly impossible to retrace the diagram without lifting your pencil from the page. But if you have a single connected diagram ($C = 1$), then it is possible redraw the diagram without lifting your pencil from the page AND without passing through a given intersection point. Moreover, the path you trace shows how to color the diagram with just two colors.

We proved this if all the intersection points are simple—one line over a previously drawn line—but the result still holds even for diagrams with multiple intersection points: just tease a complicated intersection apart into a set of simple intersections, trace and color this new diagram as before, and then argue that traceablity and two-colorability still hold when we jostle the simple intersection points back into a multiple intersection point.

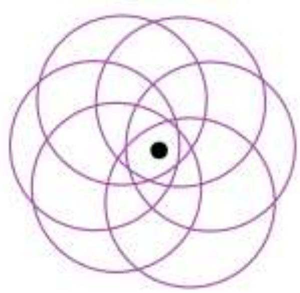


CIRCLES ON A PAGE

Suppose N circles are drawn on a page to make one connected diagram ($C=1$). If I is the total number of intersections (counted with multiplicity) together with the total count of points of tangency, then we know the number of bounded regions R on the page is given by $R=I+1$.

Thus to create a diagram with the largest or smallest count of bounded regions, we need to create the largest or smallest count of intersections points possible.

Two circles intersect at most twice. (One point of intersection is a point of tangency.) With N circles on a page, the largest count of intersection points comes from a diagram with each and every pair of circles intersecting twice. Such a diagram is possible. Here are six circles intersecting a maximal number of times, for example.



(Can you see why it behooves us to avoid multiple intersections?)

With N circles, there are $\binom{N}{2}=\frac{N(N-1)}{2}$ pairs of circles each producing two intersection points. Thus the maximal possible value for I is $N(N-1)$ and the maximal number of bounded regions created by N circles on a page is

$$R=N(N-1)+1.$$

Lining N circles in a row with $N-1$ points of tangency between them gives the least value of I and thus the least value $R=N$ for the number of bounded regions.

Question: What is the least number of bounded regions possible for a connected diagram of N circles on a page if points of tangency are to be avoided?

POLYGONS ON A PAGE

Two triangles drawn on a page can intersect six times to create 7 bounded regions. And N triangles on a page can create a maximum of

$R=6\binom{N}{2}+1=3N(N-1)+1$ bounded regions.

For N squares drawn on a page we can create a maximum of $R=4N(N-1)+1$ bounded regions. And for N convex k-gons, a maximum of $R=kN(N-1)+1$ bounded regions. (And each of these maxima are realizable. Do you see how?)

RESEARCH CORNER

Generalize this work to three-dimensional solids. For example, what is the maximal number of bounded regions formed by N spheres in space?

Explore the maximal number of bounded regions possible from a diagram of N concave k-gons on a page.

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