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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2016

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep, joyous, and real mathematical doing I'll happily mention it here.*

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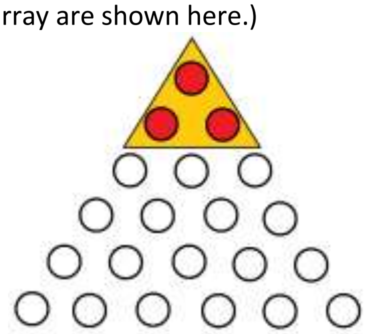
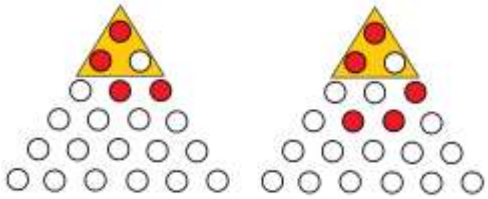
Enjoy James Propp's spectacular monthly essays at his **Math Enchantments** site: https://mathenchant.wordpress.com/. Really, truly top notch!

THE GREAT ESCAPE: Here's a classic solitaire puzzle to be played on an infinite triangular array. (Only the first six rows of the array are shown here.)



The puzzle begins with three coins placed at the apex of the array and a *move* consists of removing a coin from the array that has two empty cells just below it and replacing it with two coins in those empty cells. For example here are two moves for a start to the game.

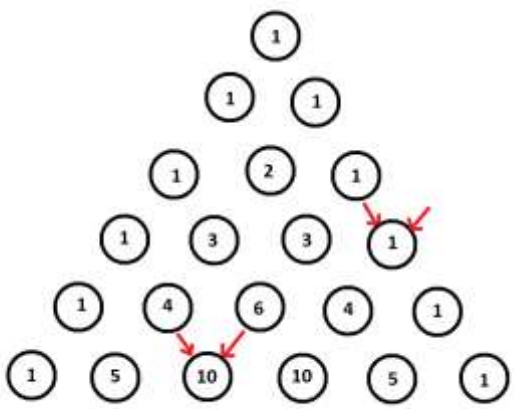


The object of the game is to empty the top three cells of the triangle of coins.

Can you do it? How many rows of the triangle do you need to accomplish the task?

LEIBNIZ' HARMONIC TRIANGLE

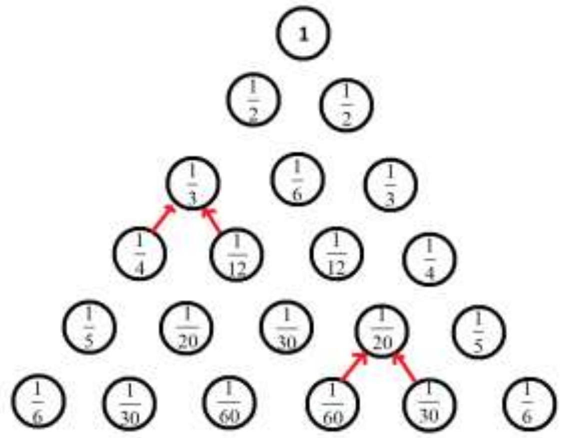
We are familiar with Pascal's Triangle.



Here the apex entry is 1 and each entry thereafter is the sum of the two entries just above it (including side entries if we regard blank positions as zero). The triangle is symmetrical.

Is there a symmetrical triangle with apex 1 and with each entry the sum of the two entries just <u>below</u> it?

There is! Take the reciprocal of each entry in Pascal's triangle and multiply it by $\frac{1}{n+1}$ where n is that entry's row number (calling the apex of the triangle row zero). This gives Leibniz' Harmonic Triangle.



Comment: The entry on the n th row of Pascal's Triangle, a places in from the left and b places in from the right ($a+b=n$) has value $\frac{n!}{a!b!}$. (See Part 3 of http://gdaymath.com/courses/permutations-and-combinations/.) The matching entry in Leibniz' Triangle is thus $\frac{a!b!}{(n+1)!}$.

The entries on the first diagonal of the triangle are

$$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots.$$

On the second diagonal we have

$$\frac{1}{1\times 2},\frac{1}{2\times 3},\frac{1}{3\times 4},\frac{1}{4\times 5},\ldots.$$

On the third diagonal we have

$$\frac{2}{1\times 2\times 3},\frac{2}{2\times 3\times 4},\frac{2}{3\times 4\times 5},\ldots.$$

Next

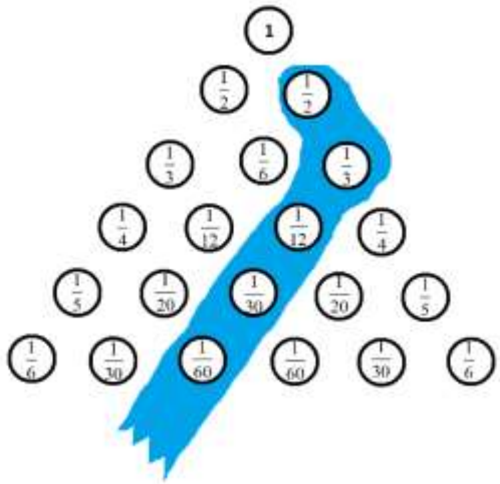
$$\frac{3!}{1\times 2\times 3\times 4},\frac{3!}{2\times 3\times 4\times 5},\frac{3!}{3\times 4\times 5\times 6},\ldots.$$

And so on.

THE INFINITE STOCKING PROPERTY

Leibniz' Harmonic Triangle has the following lovely property.

Circle any entry in the triangle. Take one "step" downwards either in a south east or south west direction to a next entry. Turn 90 degrees and then follow all the entries downwards thereafter to make an infinitely long stocking. Then the infinite sum of *entries in the leg of the stocking has value the entry in the toe.*



The diagram shows the stocking for the sum

$$\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\frac{1}{60}+\cdots=\frac{1}{2}.$$

For this example we have that $\frac{1}{2}$ is the sum of the two entries below it.

$$\frac{1}{2}=\frac{1}{3}+\left(\frac{1}{6}\right).$$

And the one-sixth is the sum of the two entries below it.

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{12}+\left(\frac{1}{12}\right).$$

And the one-twelfth is the sum of the two entries below it.

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{12}+\frac{1}{30}+\left(\frac{1}{20}\right).$$

And so on. In this way we construct the infinite sum through partial sums (and we also see that the difference between the partial sums and $\frac{1}{2}$ goes to zero).

> **EXERCISE:** What are the values of the sums
> $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$, $\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)}$, and
> $\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)\cdots(k+r)}$?

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VARIATIONS OF LEIBNIZ' HARMONIC TRIANGLE

One can place any sequence of values on the left diagonal of a triangle. The property that each entry is the sum of the two entries below it determines the entire triangle. For example, placing the fractions $1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots$ on the left diagonal gives a triangle that begins

$$\begin{array}{c}1\\ \frac{1}{3}\quad\frac{2}{3}\\ \frac{1}{5}\quad\frac{2}{15}\quad\frac{8}{15}\\ \frac{1}{7}\quad\frac{2}{35}\quad\frac{8}{105}\quad\frac{48}{105}\end{array}$$

> **CHALLENGE:** Is there are general formula for the entries of this triangle? Does the infinite stocking property hold for this triangle?

Placing the counting numbers on the diagonal gives the triangle.

$$\begin{array}{c}1\\ 2\quad -1\\ 3\quad -1\quad 0\\ 4\quad -1\quad 0\quad 0\end{array}$$

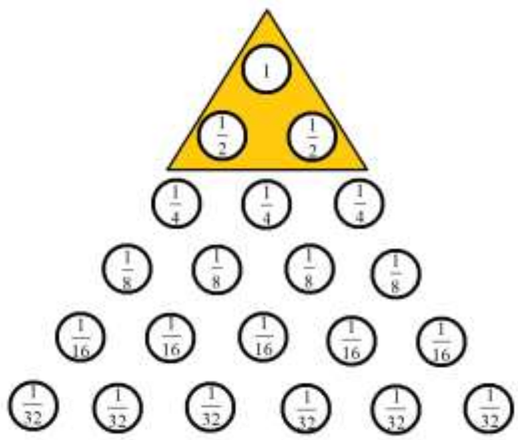
One usually does not obtain symmetrical triangles.

> **RESEARCH:** Find conditions on a sequence $a,b,c,d,e,\ldots.$ used as the left diagonal of a triangle that ensure that the resulting Leibniz' Triangle is symmetrical.
>
> **1**
> **a** _
> **b** _ _
> **c** _ _ _
> **d** _ _ _ _
> **e** _ _ _ _ _

Find conditions on the sequence $a, b, c, d, e, ...$ that ensure that the infinite stocking property holds for the triangle.

SOLVING THE OPENING PUZZLE

Using the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ as the left diagonal gives a symmetrical – and very simple - version of Leibniz' Harmonic triangle. We can use it to solve the opening puzzle.



The puzzle begins with three coins covering the cells of values 1, $\frac{1}{2}$, and $\frac{1}{2}$, a total value of $1+\frac{1}{2}+\frac{1}{2}=2$. Any move of the game removes a coin and replaces it with two coins just below it. Leibniz' Triangle is designed so that the revealed entry matches the sum of the values covered by the two new coins. This ensures that the "total value" of the game never changes.

As the game is played, the sum of values under all the coins on the board is sure to equal 2.

The goal of the game is to have no coins in the top three positions of the array, that is, to produce a configuration of coins covering only cells of value $\frac{1}{4}$ or less.

There are three positions of value $\frac{1}{4}$, four of value $\frac{1}{8}$, five of value $\frac{1}{16}$, and so on. I am curious about the value of the infinite sum

$$3\times\frac{1}{4}+4\times\frac{1}{8}+5\times\frac{1}{16}+6\times\frac{1}{32}+\cdots,$$

the total value of all the cells under the top three positions of the triangle.

We can evaluate this sum by using the values of these sums:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2}$$

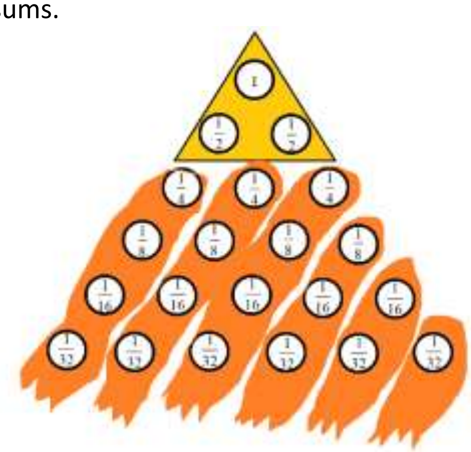
$$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=\frac{1}{4}$$

and so on.

[And to get these, let $s=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$.

Then $2s=1+\frac{1}{2}+\frac{1}{4}+\cdots=1+s$, giving $s=1$. The remaining sums follow by dividing by two throughout.]

Our sum is an infinite collection of these sums.



$$3\times\frac{1}{4}+4\times\frac{1}{8}+5\times\frac{1}{16}+6\times\frac{1}{32}+\cdots=$$

$$= \left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)$$
$$+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)$$
$$+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)$$
$$+\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\right)$$
$$+\left(\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots\right)$$
$$+\left(\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\cdots\right)$$
$$+\cdots$$
$$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$
$$=\frac{1}{2}+\frac{1}{2}+1$$
$$=2.$$

This shows that if we place a coin on each and every cell of value $\frac{1}{4}$ or less, then the total value of that configuration will be 2, on the nose. Thus if we leave any cells uncovered, the value of the configuration will be less than 2.

And recall that 2 is the total value of the game at any stage of play.

So we have an interesting pickle here. At any stage of play there will be only finitely many coins on the board. The sum of values covered by the coins at any stage of play must be 2. To get a sum of values equal to 2 away from the top three positions of the array we must have an infinite number of coins in play. This is not possible.

It is thus impossible to solve the opening puzzle in a finite number of moves. We cannot escape the top three positions at the apex of the triangle!

EXERCISE: Suppose we are god-like and can play beyond the end of time to conduct an infinite number of moves. Why is it still not possible to solve the puzzle? (How many coins can there be on the left edge of the array?)

CHALLENGE:
For those who think calculus …. Is the following argument for computing our infinite sum valid?

The geometric series formula is

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4\cdots.$$

Differentiate to get

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+\cdots.$$

Put $x=\frac{1}{2}$ *to see*

$$4=1+1+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots$$

giving the sum we seek, namely,

$$\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots=2.$$

RESEARCH CORNER

Pascal's Triangle is full of astounding patterns: the entries of its rows sum to powers of two, the alternating entries of its rows sum to zero, the rows match powers of eleven, and so on. (Again see part 3 of http://gdaymath.com/courses/permutations-and-combinations/.)

Explore patterns in Leibniz' Harmonic Triangle. Anything to be said about row sums? Alternating row sums? Links to a version of the Binomial Theorem?

Do variations of Leibniz' Triangle lead to any interesting new summation formulas?

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