CURRICULUM INSPIRATIONS: www.maa.org/ci
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Innovative Online Courses: www.gdaymath.com
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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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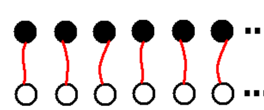
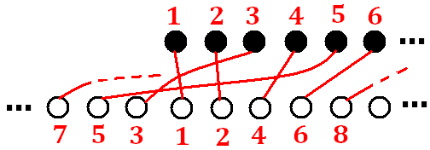
Check out John Miller's music of the polygons: www.polygonjazz.com . Stunningly beautiful (even in just watching the video). And there is follow-up math too. Check out: www.polygonflux.com/math-minded.html

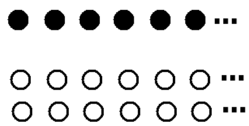
PUZZLER: In the picture below each black dot represents a person (there are infinitely many of them) and each white dot a dog (and there are infinitely many of them too). Each person is holding a leash to one dog and each dog is leashed to one person.



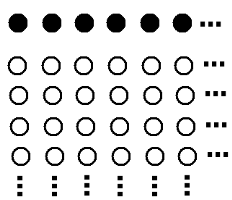
In the next picture we have another infinite row of people, but a doubly infinite row of dogs. The diagram shows, surprisingly, that it is still possible to leash each person to a dog and each dog to a person!



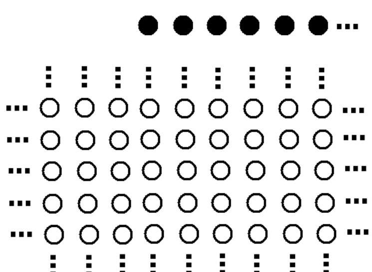
a) Show how to leash a single infinite row of people to two infinite rows of dogs. (Make sure each person will be leashed to a dog and each dog to a person.)



b) Show how to leash a single infinite row of people to an infinite number of infinite rows of dogs!



c) Show how to leash a single infinite row of people to a full two-dimensional infinite array of dogs!

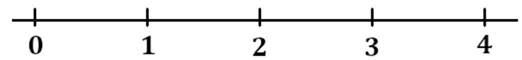


INTEGERS ON THE NUMBER LINE

Here's a strange question:

How much space do the integers take up on the number line?

Each integer is a point and so takes up no space.



But there are infinitely many of these integer points so maybe those points "add up" to some total amount of space? After all, there is one unit of length between 0 and 1 on the number line and all the points between 0 and 1, each of zero width, do add up to one unit of length – somehow.

Hmm. But I do feel that the integer points, because they are "spaced apart," don't add up to any meaningful length overall.

What to do?

Here's one way to think about how much space a set of points take up on the number line: Cover them by lengths of ribbon!

First note that the decimal $0.1111...$ is the fraction $\frac{1}{9}$. (To see this, note that if $0.1111... = x$, then $10x = 1.111.... = 1 + x$ from which it follows $x = 1/9$.)

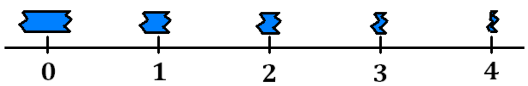
So this means that if I have a piece of ribbon 0.1 units long, and a piece of ribbon 0.01 units long, and a piece of ribbon 0.001 units long, and so on - an infinite number of pieces of ribbon following this pattern of decreasing lengths - then the total amount of ribbon I possess is only

$$0.1 + 0.01 + 0.001 + \cdots = 0.1111\cdots = \frac{1}{9}$$

of a unit. It's not much ribbon.

Now let's use the ribbon to cover the integer points on the number line. (Just for ease, let's just focus on the positive half of the number line, zero upwards.)

First cover the point "0" with the 0.1 length of ribbon. Then cover "1" with the 0.01 length. And then "2" with the 0.001 length. And so on, all the way down the number line.



Certainly each integer point is covered by ribbon and I've used a total of $\frac{1}{9}$ of a unit of ribbon. Thus I can declare, for sure, that:

The integers take up no more than $1/9$ *of a unit of space on the positive number line.*

But now imagine I based this reasoning on the decimal $0.010101010... = \frac{1}{99}$. (You can see that this equals one-ninety-ninth by giving it a name and multiplying it by 100.) That is, if I used ribbons of lengths 0.01, and 0.0001, and 0.000001, and so on, I would declare instead:

The integers take up no more than $1/99$ *of a unit of space on the positive number line.*

And if I focused on the quantity $0.001001001... = \frac{1}{999}$, I would declare:

The integers take up no more than $1/999$ *of a unit of space on the positive number line.*

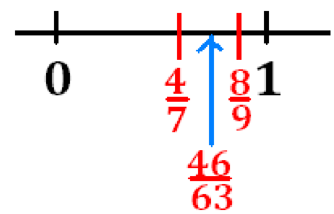
So the amount of space of space taken up by the integers on the positive half of the number line is less than any of the numbers in this list:

$$\frac{1}{9}, \frac{1}{99}, \frac{1}{999}, \frac{1}{9999}, \ldots$$

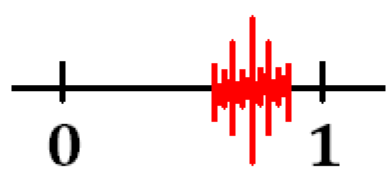
I can only conclude that the integers take up zero space.

FRACTIONS ON THE NUMBER LINE

For any two fractions on the number line there is certainly a fraction between them – just take their average for example. (For $\frac{a}{b}$ and $\frac{c}{d}$, we see that $\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)$ is a fraction too.)



In fact, we can keep filling up the space between any two fractions with infinitely many fractions.



And it seems that the fractions fill up the entire number line. So if I asked:

How much space do the fractions take up on the number line?

the answer seems to be: *All of it.*

Shockingly, this answer is wrong! We shall prove that **the fractions take up zero space on the number line.** We'll use the same ribbon idea as we did for the integers to do this.

Recall we covered each integer, in turn, with a piece of ribbon. As the total length of ribbon used was $1/9$ of a unit long, or $1/99$ of a unit long, or $1/999$ of a unit long, and so on, we saw that the total amount of space taken up by the integers is no more than $1/9$ of a unit, actually no more than $1/99$ of a unit, actually no more than $1/999$ of a unit, and so on. This led us to conclude that the total amount of space they take has to be zero.

In fact, this argument would work for any set of points on the number line, as long as we can list those points in turn: we need a first point to cover, a second point to cover, a third point to cover, and so on. And even if the points are spaced so that the ribbons overlap a little bit in places, no matter! The total amount of ribbon used is still $1/9$ or $1/99$ or $1/999$ and so on, and so the total amount of space taken by those points is still no more than $1/9$ of a unit, no more than $1/99$ of a unit, etc.

So … can we apply the ribbon argument to positive fractions?

PROBLEM: What is the first positive fraction? What is the second positive fraction? What is the third?

These questions pose a serious problem: *What is the first fraction to the right of zero?* Answer: There is no first fraction! Whatever you think might be first positive fraction, just halve it and you have a fraction closer to zero proving you were wrong. There is no first fraction to the right of zero!

But no one says we have to put the fractions in a list in increasing order. Any list that runs through all the fractions will do. We'll call the first fraction on such a list the "first fraction," the second fraction on the list the "second fraction," and so on, irrespective of where those fractions actually sit on the number line.

So … **Is there some way to list the fractions if we don't care about preserving their order?**

Sticking with the positive half of the number line, we can certainly organize the positive fractions in a two-dimensional table:

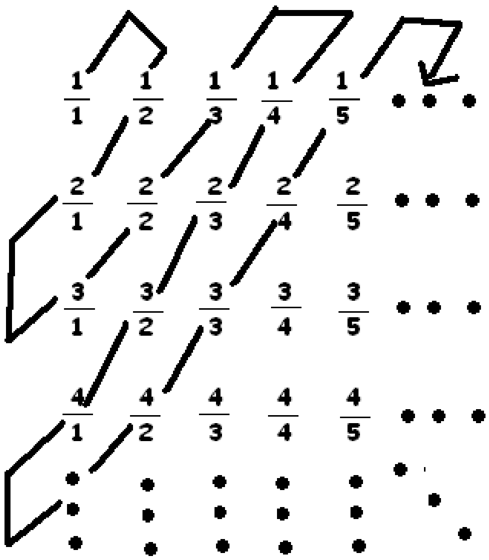
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\cdots$
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\cdots$
$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

This table is inefficient: the fraction $\frac{2}{1}$ is repeated as $\frac{4}{2}$, for example, and $\frac{3}{5}$ is repeated as $\frac{30}{50}$ later on in the table. No matter. It is nonetheless clear that every fraction is indeed listed in the table – somewhere.

The trouble with this table is that it is not a single list. Hmm.

Can we use the infinite two-dimensional array of fractions to create a single one-dimensional list of fractions?

German mathematician Georg Cantor (1845-1918) wondered this too. And one day he had an epiphany: chase through the diagonals of the table!



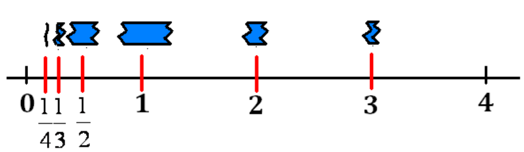
As we go along the diagonals we'll encounter repeat fractions. Not a problem: just ignore the repetitions! Thus we get a list of the rationals that begins:

$$\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{3}{1},\ \ \frac{1}{3},\frac{1}{4},\frac{2}{3},\frac{3}{2},\frac{4}{1},\frac{5}{1},\ \ \frac{1}{5},\frac{1}{6},\cdots$$

(I've left spaces to show where I ignored repeats.)

This list will indeed include every possible positive rational number: every positive fraction appeared in the table and so every positive fraction will be encountered by following the diagonals.

So that's it: Just cover the fractions in this order with pieces of ribbon:



We conclude: **Indeed, the fractions occupy no space on the number line!**

IRRATIONALS ON THE NUMBER LINE

Any number that is not rational is, by definition, irrational. Since the rationals take up no space on the number line we can only conclude that the number line is essentially nothing but points representing irrationals!

We hear that numbers like $\sqrt{2}$ and π are irrational. (How does one know? Have you ever proved that $\sqrt{2}$ is not a fraction? Have you checked its decimal expansion all the way out to make sure it never repeats?). And we feel that such numbers are exceptions. This is probably because we only ever seem to focus on integers and fractions in our thinking. But it turns out that the irrationals are the majority and the integers and the fractions are the rarities. (Whoa! My mathematical universe has just been upturned!)

Research Corner:

a) Explain why numbers that are given by infinite repeating decimals, such as $0.\overline{19}$ and $0.72\overline{043}$ and $0.5\overline{0}$, are sure to be rational.

b) Consider the number with decimal expansion $0.1\,|\,2\,|\,3\,|\,4\,|\,5\,|\,6\,|\,7\,|\,8\,|\,9\,|\,10\,|\,11\,|\,12\,|\,13\,|\cdots$ What is its value? It is rational! (It is okay to put more than ten pips in a decimal position: some carrying will occur. For example $0.3\,|\,12\,|\,8 = 0.4\,|\,2\,|\,8$.)

c) Is $0.1\,|\,4\,|\,9\,|\,16\,|\,25\,|\,36\,|\cdots$ rational? If so, what is its value?

d) Is $0.1^k\,|\,2^k\,|\,3^k\,|\,4^k\,|\cdots$ always a repeating decimal once all the carries have occurred?

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