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WOWZA COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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For teaching problem solving in the mathematics classroom check out the ever-growing MAA Curriculum Inspirations resources at www.maa.org/ci. Videos, essays, strategies, and, best of all, just plain, neat, cool, interesting math!

OPENING PUZZLE:
By a *compositional square root* of a function f we mean a function h such that $h(h(x)) = f(x)$ for all inputs x.

For example, a compositional square root of the function f given by $f(x) = x+1$ is h given by $h(x) = x + \frac{1}{2}$.

A compositional square root of f instead given by $f(x) = x^2$ is h given by $h(x) = x^{\sqrt{2}}$.

a) Are the compositional roots given in these two examples each unique?

b) What is a compositional square root of f given by $f(x) = x^2 + 1$?

COMPOSITIONAL SQUARE ROOTS

Given a function f it is usually very difficult, if not nigh on impossible, to give a formula for a function h which satisfies $h(h(x)) = f(x)$ for all, or perhaps just some range, of inputs x. But on occasion it can be done. For example, a compositional square root of the function defined by $f(x) = x+1$ is indeed given by $h(x) = x + \frac{1}{2}$. But this is not its only one!

SURPRISE: *The function* h *given by*

$$h(x) = \begin{cases} 2x + \frac{1}{3} - k & \text{if } x \in [k, k+\frac{1}{3}] \text{ for some integer } k. \\ \frac{1}{2}x + \frac{5}{6} + \frac{k}{2} & \text{if } x \in [k+\frac{1}{3}, k+1] \text{ for some integer } k. \end{cases}$$

is also a compositional square root of $f(x) = x+1$.

Check: If $k \le x \le k + \frac{1}{3}$, then

$$2k + \frac{1}{3} - k \le 2x + \frac{1}{3} - k \le 2k + \frac{2}{3} + \frac{1}{3} - k.$$

That is, $k + \frac{1}{3} \le h(x) \le k+1$, and so:

$$\begin{aligned} h(h(x)) &= \frac{1}{2}h(x) + \frac{5}{6} + \frac{k}{2} \\ &= \frac{1}{2}\left(2x + \frac{1}{3} - k\right) + \frac{5}{6} + \frac{k}{2} \\ &= x+1. \end{aligned}$$

If, instead, $k + \frac{1}{3} \le x \le k+1$, then

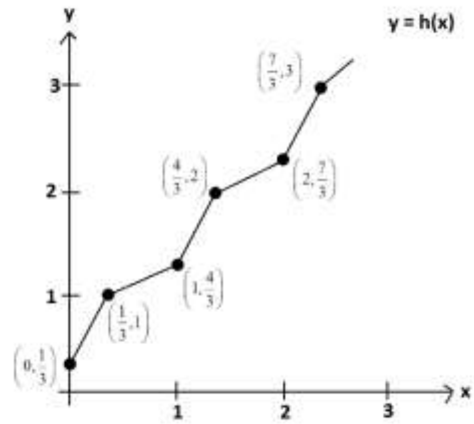
$$\frac{k}{2} + \frac{1}{6} + \frac{5}{6} + \frac{k}{2} \le h(x) \le \frac{k}{2} + \frac{1}{2} + \frac{5}{6} + \frac{k}{2}.$$

That is, $k+1 \le h(x) \le k+1+\frac{1}{3}$, and so

$$\begin{aligned} h(h(x)) &= 2h(x) + \frac{1}{3} - (k+1) \\ &= 2\left(\frac{1}{2}x + \frac{5}{6} + \frac{k}{2}\right) + \frac{1}{3} - k - 1 \\ &= x+1. \end{aligned}$$

Whoa!

Here's a graph of $y = h(x)$:

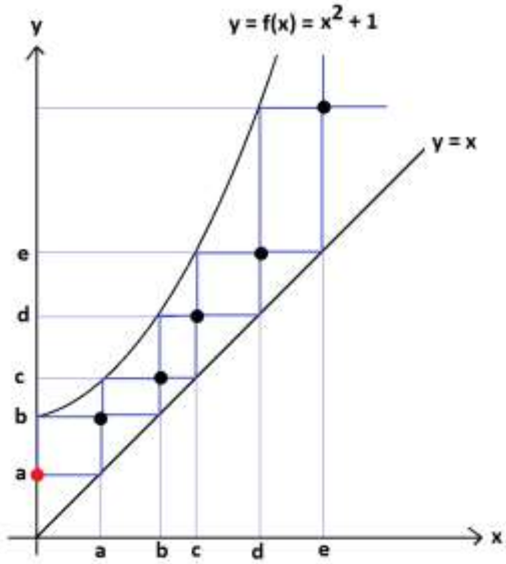


We certainly see that compositional square roots need not be unique (and need not differ only by a minus sign!).

CONSTRUCTING COMPOSITIONAL SQUARE ROOTS

Here's how to construct a compositional square root to the function f given by $f(x) = x^2 + 1$, at least geometrically. (I used this technique to construct the unusual answer in the previous section.)

Draw the graphs of $y = f(x)$ and $y = x$ and choose a point on the y-axis between the two y-intercepts of the graphs, say $(0, a)$. From that point construct rectangles between the two graphs as shown. These identify a series of points (a, b), (b, c), (c, d), … between the two curves.



To start defining a function h set:

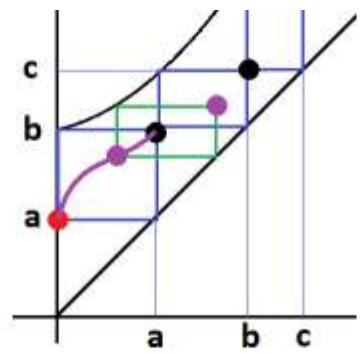
$$h(0) = a$$
$$h(a) = b$$
$$h(b) = c$$
$$h(c) = d$$

and so on. Observe:

$$h(h(0)) = h(a) = b = f(0)$$
$$h(h(a)) = h(b) = c = f(a),$$
$$h(h(b)) = h(c) = d = f(b)$$

and so on.

To define h for inputs between $x = 0$, $x = b$, $x = c$, etc., draw any curve that connects the corners of the first rectangle, that is, connects the points $(0, a)$ and (a, b), and represents the graph of a strictly increasing function.



Then, with the aid of a rectangle, each point on that curve defines a point on a curve in the next rectangle. And each point on the curve in that rectangle defines a point on a curve in the next rectangle, and so on.

We now have the graph of a function h with the property that $h(h(x)) = f(x)$.

This construction shows that there are usually an infinitude of possible compositional square roots to a given function.

RESEARCH CORNER

Of course one wonders if the previous construction could ever have problems.

1. In this essay we constructed a compositional square root of $f(x) = x^2 + 1$ for the range of positive inputs. Can we extend the construction to negative inputs too?

2. In our construction of a compositional root of $f(x) = x^2 + 1$ I suggested we draw the graph of a strictly increasing function in

the first rectangle. Is the "strictly increasing" condition necessary?

3. Would the rectangle method run into trouble, in general, if the graphs of $y = f(x)$ and $y = x$ intersect? (Can you construct a second compositional square root of $f(x) = x^2$?)

4. Must the graph of $y = f(x)$ be continuous for the rectangle method to work? For example, the function $f(x) = \lceil x \rceil$ (round x up to the next integer) is its own compositional square root. Can the rectangle method be modified to construct another compositional root of it?

5. For those who know calculus: Is $h(x) = x + \frac{1}{2}$ the only differentiable compositional square root of $f(x) = x + 1$?

6. Must every function $f : \mathbb{R} \to \mathbb{R}$ have a compositional square root?

7. Is there a geometric method for constructing compositional cube roots? Compositional fourth roots?

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