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WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MARCH 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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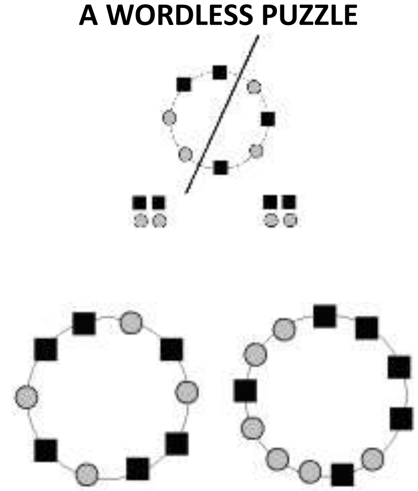
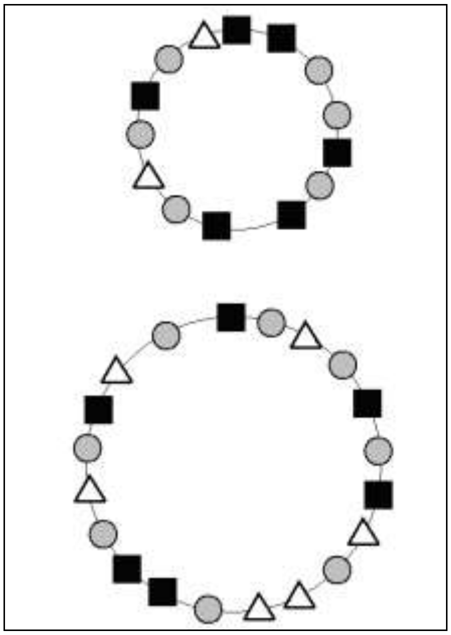
Mathematician and curriculum creator Scott Baldridge has a new website: http://scottbaldridge.net/channels-page/ See television shows on research mathematics discussed at a high-school level, blogs on engineering the new Common Core Eureka Math/EngageNY curriculum, and lots of cool pieces and videos on making mathematics accessible and real to all of all ages!

A WORDLESS PUZZLE





A COUNTERPOINT PUZZLE

I flip a coin. I first toss a TAIL, but after some time tossing I notice that actually over 80% of my tosses were HEADS. Must, at some point, exactly 80% of my tosses have been HEADS? Must, at some point, exactly 75% of my tosses have been HEADS? Exactly 70% of my tosses?

Comment: This second puzzle has recently been making "the rounds." A version of it first appeared as problem 1 in the 2004 Putnam Mathematical Competition.

THE DISCRETE INTERMEDIATE-VALUE THEOREM

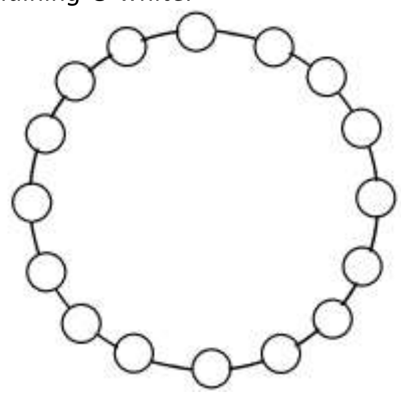
Here's a simple idea:

Suppose I walk on the number line, starting at position 5 *and randomly take unit steps to the left and to the right. If, after a while, I end up at position* 25*, then it must be the case that I stood on each and every number between* 5 *and* 25 *at some point.* (I might even have stood on the same number more than once.)

It is impossible, for example, to miss stepping on position 17 : doing so requires, at some point, being at location 16 or lower and stepping over to a position 18 or higher. With unit steps, this is impossible.

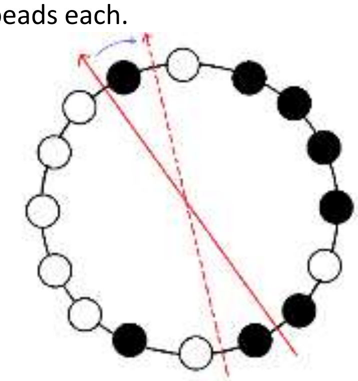
Here's a lovely application of this counting observation:

THE NECKLACE THEOREM (for 16 beads)
Here are 16 *beads on a necklace. Color any* 8 *of them black and leave the remaining* 8 *white.*



No matter the coloring pattern you choose, it is always possible to cut the necklace in just two places to obtain two necklace halves each containing exactly 4 *black beads and* 4 *white beads.*

Reason:
Draw an arrow through the center of the necklace circle, passing between beads and dividing the necklace into two halves with eight beads each.



Suppose there are b black beads on the left half of the necklace as defined by the arrow. (Consequently, there are $8-b$ beads on the right half.)

If $b = 4$, then we have, by luck, found a diameter that splits the necklace into two halves, each with four beads of each color.

If not, rotate the arrow one notch over. In doing so, the left half of the necklace picks up a bead and loses a bead.

If it picks up a black bead and loses a white bead, the count of black beads to the left of the arrow changes from b to $b+1$.

If it picks up a white bead and loses a black bead, the count of black beads to the left of the arrow changes from b to $b-1$.

If it picks up and loses beads of the same color, the count b does not changed.

Now march the arrow eight places over until it lands in its original position, but now pointing in the opposite direction. The count of black beads to its left is $8-b$. (The current "left" is the original "right.") If b is a number lower/higher than 4, then $8-b$ is a number higher/lower than 4. As these numbers change at most one unit at a time, there must be an intermediate arrow position with the count of back beads to its left equal to 4. That is, the arrow must pass through a position that divides the necklace into two halves with equal numbers of black and white beads in each half.

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Of course there is nothing special about the number 16 here. In general, we have:

THE NECKLACE THEOREM
Any necklace possessing $2N$ white beads and $2M$ black beads can be cut into contiguous two halves so that each half contains N white and M black beads.

PERCENTAGES OF COIN TOSSES

The second puzzle, the coin-tossing puzzle, also has the feel of the Intermediate-Value Theorem in play:

In order to move from 0% heads (which was the case after our first toss) to a number above 80% heads, surely we move through each of the percentage values 80%, 75%, and 70%?

The answer, surprisingly, is YES, YES, and NO!

Example: Suppose my first toss is indeed TAILS, but I obtain HEADS each and every toss thereafter. Here's a table showing how the percentage of heads grow:

T	**0/1**	**=**	**0%**
TH	**1/2**	**=**	**50%**
THH	**2/3**	**=**	**67%**
THHH	**3/4**	**=**	**75%**
THHHH	**4/5**	**=**	**80%**
THHHHH	**5/6**	**=**	**83%**

We skipped over 70%! (But did pass through 75% and 80%.)

CLAIM: *For any sequence of coin tosses, starting with first toss TAILS and ending with over 80% HEADS, there must have been some intermediate sequence possessing exactly 80% HEADS.*

Reason: Let's record a toss of a HEADS with the numerical value 1 and a toss of TAILS with 0. Let ε_n denote the value of the nth toss. (So $\varepsilon_n = 1$ if the nth toss is HEADS and $\varepsilon_n = 0$ if the nth toss is a TAILS.)

We are told that $\varepsilon_1 = 0$.

The sum $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$ equals the number of HEADS among the first n tosses, and $\dfrac{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n}{n}$ is the fraction of HEADS among the first n tosses.

We want to show that at least one of these fractions equals $\frac{4}{5}$ (that is, 80%). We do know that the fraction sequence:

$$\frac{\varepsilon_1}{1}=0$$
$$\frac{\varepsilon_1+\varepsilon_2}{2}$$
$$\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3}{3}$$
$$\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{4}$$
$$\vdots$$

starts off with the value zero and ends with a value greater than $\frac{4}{5}$.

Suppose, to the contrary, none of these fractions equals $\frac{4}{5}$. What goes mathematically awry with this bold, contrary assumption?

There must be some fraction $\frac{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k}{k}$ in the list under $\frac{4}{5}$ with the next fraction in the list $\frac{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k+\varepsilon_{k+1}}{k+1}$ greater than $\frac{4}{5}$.

And for these fractions, it must be that $\varepsilon_{k+1}=1$.

(If $\varepsilon_{k+1}=0$, then the fraction $\frac{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k+0}{k+1}$ is smaller than $\frac{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k}{k}$, not larger.)

So we have the inequalities:

$$5\left(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k\right)<4k$$

and

$$5\left(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k\right)+5>4k+4\,.$$

It follows that $5\left(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_\infty\right)$ is an integer strictly between $4k-1$ and $4k$. This is indeed absurd, and so our assumption that these fractions miss the value $\frac{4}{5}$ is wrong.

EXERCISE: Repeat this proof to show that the sequence of fractions must also pass through the value $\frac{3}{4}$ (and the value $\frac{2}{3}$ and the value $\frac{1}{2}$!)

The most general result is:

COIN TOSSING THEOREM:
Any sequence of coin tosses, starting with first toss TAILS and ending with a fraction q of tosses being HEADS, must have intermediate sequences with $\frac{1}{2}$ HEADS, $\frac{2}{3}$ HEADS, $\frac{3}{4}$ HEADS ,and so on all the way up to fraction $\frac{a-1}{a}$ HEADS, where $\frac{a-1}{a}$ is the largest fraction with numerator one less than denominator less than or equal to q.

I wonder how many students taking the 2004 Putnam Exam (incorrectly) answered question 1 by citing a discrete Intermediate-Value Theorem? (I was tempted to do so too when I first told this problem!)

RESEARCH CORNER

The Puzzle-Without-Words suggests working with necklaces possessing beads of more than two colors.

Question: *Given 8 rouge beads, 8 mauve beads, and 8 lilac beads arranged in a circular necklace, is there sure to be a diameter that divides the necklace into two halves each containing four beads of each color?*

The following alternative variation of necklace cutting is a proved result in mathematics:

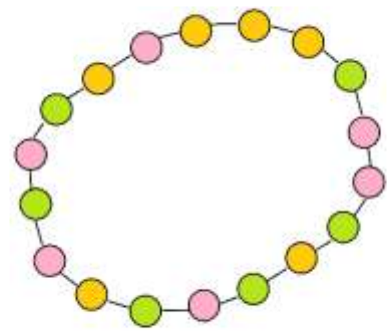
THE MULTI-COLORED NECKLACE THEOREM

Suppose a necklace contains beads of k different colors, and there are $2N$ beads of each color.

Case k is even: It is possible to cut the necklace in k places and share the resulting k pieces between two people so that each person obtains N beads of each color. (Fewer than k cuts might not be possible.)

Case k is odd: It is possible to cut the necklace in $k+1$ places and share the resulting $k+1$ pieces between two people so that each person obtains N beads of each color. (Fewer than $k+1$ cuts might not be possible.)

For example, the necklace below has beads of $k=3$ colors. According to the theorem, it is possible to cut the string in just 4 places to get four necklace pieces that can be divvied between two people with each person ending up with three beads of each color. Do you see how?



The version of the theorem with $k=2$ and $2N=8$ is our original necklace theorem.

RESEARCH:

I don't know a simple proof of the multi-colored necklace problem. In fact, the only proof of it I know is surprisingly technical and advanced, drawing on mighty sophisticated, scary, high-level mathematics!

Surely there must be a relatively straight-forward proof of the result?
Care to find one? (At least for the three-color version?)

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