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CURIOUS MATHEMATICS FOR FUN AND JOY

JUNE 2020

THIS MONTHS' PUZZLER:

a) Which two positive integers summing to 2718 have the largest product?

b) Which three positive integers summing to 2718 have the largest product?

c) Which ten positive integers summing to 2718 have the largest product?

d) Which set of positive integers, of any size, summing to 2718 has the largest product?

For those who know calculus …

e) Which set of positive real numbers summing to 2718 has the largest product?

USING THE A-G INEQUALITY

In the March 2020 essay, we discussed the famous arithmetic mean-geometric mean inequality (A-G inequality) and put it to good use. This essay brings us the opportunity to wield its power one more time.

Recall that if a_1, a_2, …, a_N is a set of positive numbers, then their arithmetic mean

$$A = \frac{a_1 + a_2 + \cdots + a_N}{N}$$

and their geometric mean

$$G = \sqrt[N]{a_1 \cdot a_2 \cdot \cdots \cdot a_N},$$

satisfy

$$A > G$$

unless all the terms are equal ($a_1 = a_2 = \cdots = a_N$), in which case $A = G$.

We always have

$$a_1 + a_2 + \cdots + a_N = \overbrace{A + A + \cdots + A}^{N \text{ copies}},$$

but

$$a_1 + a_2 + \cdots + a_N \ge \overbrace{G + G + \cdots + G}^{N \text{ copies}}$$

with equality only if all the terms are equal.

Having read the March 2020 essay, we know how to solve puzzles a) and b).

a) From $a + b = 2718$ it follows that

$$2718 = a + b \ge \sqrt{ab} + \sqrt{ab}$$

and so

$$\sqrt{ab} \le 1359$$

or

$$ab \le 1846881.$$

This is the largest possible product and it is attained (that is, we have equality in this inequality) if $a = b$. So choose $a = b = 1359$.

b) From $a + b + c = 2718$ it follows that

$$2718 = a + b + c \ge \sqrt[3]{abc} + \sqrt[3]{abc} + \sqrt[3]{abc}$$

and so

$$\sqrt[3]{abc} \le 906$$

or

$$abc \le 743677416.$$

This is the largest possible product and it is attained by choosing $a = b = c = 906$.

The same reasoning shows that for part c) we'd obtain the maximal product from a collection of ten positive numbers summing to 2718 by choosing ten numbers of the same value 271.8. One snag: the question only allows for integers.

Hmm.

We are not allowed to work with

$$\begin{aligned} 2718 = {} & 271.8 + 271.8 + 271.8 \\ & + 271.8 + 271.8 + 271.8 \\ & + 271.8 + 271.8 + 271.8 \\ & + 271.8 \end{aligned}$$

But we might suspect that working with the "closest" integer version of this might give the maximal product of summands among integer summands:

$$\begin{aligned} 2718 = {} & 271 + 271 + 272 \\ & + 272 + 272 + 272 \\ & + 272 + 272 + 272 \\ & + 272 \end{aligned}$$

Can we prove this?

Lemma 1: *Suppose a and b are positive integers and $b \ge a + 2$. Then*

$$(a+1)(b-1)$$

is larger than ab.

Proof:

$$\begin{aligned} (a+1)(b-1) &= ab + b - a - 1 \\ &\ge ab + 1 \\ &> ab \end{aligned}$$

Aha! So if we write 2718 as a sum of ten positive integers and two of those integers a and b differ by two or more (say, $b \geq a+2$), then replace $a+b$ in the sum by $(a+1)+(b-1)$. The sum has not changed, but the product of the summands has increased.

So in writing 2718 as a sum of ten positive integers with the product of those integers as large as possible, it follows that no two summands differ by two or more. We must have a sum of the form

$$2718 = a + \cdots + a + \overbrace{(a+1) + \cdots + (a+1)}^{k \text{ terms}}.$$

From $2718 = 10a + k$ we see $k = 8$ and $a = 271$, giving the sum we first wrote down.

> **Challenge:** Suppose we wish to write a positive integer S as a sum of N positive integers, with no two summands differing by two or more. When can it be done? When it can be done, is the expression essentially unique?

AVOIDING THE A-G INEQUALITY

With lemma 1 in hand we now realize that we need not make use of the A-G inequality at all in order to answer puzzles a), b), c).

Consider the general question:

Of all the ways to write an integer S as a sum of N positive integers (assuming $S \geq N$), which gives the largest possible product of summands?

As there are only a finite number of ways to write a number as a sum of N positive integers, a maximal value for the product of summands exists.

By lemma 1, no expression with two summands that differ by more than one can yield a maximal product. So the optimal expression must be of the form

$$S = a + \cdots + a + (a+1) + \cdots + (a+1).$$

And by the challenge, with $S \geq N$, there is essentially only one way to write S as such a sum. (Did you think through $2+2+3+\cdots+3$ versus $3+\cdots+3+4$?)

So the maximal product of summands occur when we write S as a sum of N terms all "as close as possible" to being equal.

For puzzles a), b), and c), the solutions must therefore indeed be given by

$$2718 = 1359 + 1359$$

$$2718 = 906 + 906 + 906$$

$$2718 = 271 + 271 + 272 + \cdots + 272$$

NO RESTRICTIONS ON THE COUNT OF SUMMANDS

Consider problem d) now. We wish to write 2718 as a sum of positive integers, any number of them, but choose the (an?) expression that gives the largest possible product of the summands.

We know such expression is a product of the form
$$2718 = a + \cdots + a + (a+1) + \cdots + (a+1)$$
composed of at most just two numbers, a and $a+1$.

Should those two numbers be 1 and 2? Or2 and 3? Or 3 and 4? Or 92 and 93?

A decomposition into 1s and 2s is no good. If there are two 1s replace them with a 2 and increase the product. If there is a 1 and a 2, replace them with a 3 and increase the product again. If there are only 2s, replace $2+2+2$ with $3+3$ and increase the product yet again.

We see that an expression composed of 2s and 3s, with at most two 2s, is better than an expression composed of 1s and 2s.

For an expression composed of 3s and 4s, each 4 may as well be replaced 4 with 2+2 to obtain an expression of 2s and 3s. But then if there are more than two 2s (that is, more than one 4 to begin with), replace $2+2+2$ with $3+3$ again.

We see that an expression composed of 2s and 3s, with at most two 2s, is better than an expression composed of 3s and 4s.

In an expression composed of 4s and 5s, replace each 5 with $2+3$ and increase the product. Also replace each 4 with $2+2$ and we're back to an expression composed of 2s and 3s with at most two 2s.

It seems that 2s and 3s are the way to go!

Lemma 2: *For any number $n \geq 5$ we have that $(n-3)\times 3$ is strictly greater than n.*

Proof:

$$\begin{aligned} 3n-9 &= 2n+n-9 \\ &\geq 10+n-9 \\ &> n \end{aligned}$$

Lemma 2 shows that if a number $n \geq 5$ appears in a summand, we can replace n by $(n-3)+3$ and increase the product of the summands.

Putting together all the pieces of the observations just made we conclude

The expression that writes 2718 as a sum of positive integers that gives the greatest product of summands must be composed of 2s and 3s with at most 2s.

As 2718 is a multiple of three, writing it as a sum of nine-hundred-and-six 3s is the expression with maximal product of summands.

Challenge: Argue that every positive integer is either a multiple of three, or 2 more than a multiple of three, or 2+2 more than a multiple of three. Thus argue that every positive integer can be expressed, essentially uniquely, as a sum of 2s and 3s with a most two 2s.

FROM INTEGERS TO REAL NUMBERS

Warning: Calculus!

If we write $2718 = a_1 + a_2 + \cdots + a_N$ as a sum of N positive real numbers, then the A-G inequality tells as that

$$a_1 \cdot a_2 \cdot \cdots \cdot a_N$$

is less than or equal to $\left(\frac{2718}{N}\right)^N$ with us reaching this upper bound by choosing each a_i equal to $\frac{2718}{N}$.

$$2718 = \frac{2718}{N} + \frac{2718}{N} + \cdots + \frac{2718}{N}$$

Each value $\frac{2718}{N}$ is likely not an integer, but this is not an issue for puzzle e).

We must ask:

Which value of N gives the largest value to $\left(\frac{2718}{N}\right)^N$?

To do this, let's look at the curve of $y = \left(\frac{2718}{x}\right)^x$, defined on all positive real values x and find its maximum value. Then we can restrict x to just integer values and see what we can learn for the task at hand.

Now $y = \left(\frac{2718}{x}\right)^x$ has a maximum value at some value x for which $y' = 0$. To differentiate the function, let's first "hit both sides" with a log. We obtain

$$\ln(y) = x\ln\left(\frac{2718}{x}\right).$$

Differentiating gives

$$\begin{aligned}\frac{y'}{y} &= \ln\left(\frac{2718}{x}\right) + x \cdot \frac{x}{2718} \cdot \frac{-2718}{x^2} \\ &= \ln\left(\frac{2718}{x}\right) - 1\end{aligned}$$

For y' to equal zero, we need $\frac{2718}{x} = e$. That is, the maximum must be occurring at $x = \frac{2718}{e} \approx 999.896$. (One can check that $y' > 0$ for value $0 < x < \frac{2718}{e}$ and that $y' < 0$ for $x > \frac{2718}{e}$, so that this is indeed a maximum and the maximum is unique.)

Now if x is restricted to being an integer, then the maximum value of $\left(\frac{2718}{N}\right)^N$ occurs either for $N = 999$ or $N = 1000$.

With computer software, we see

$$\left(\frac{2718}{999}\right)^{999} \approx 1.775 \times 10^{434}$$

and

$$\left(\frac{2718}{1000}\right)^{1000} \approx 1.776 \times 10^{434}.$$

Thus $N = 1000$ gives the maximal value and we need to write 2718 as a sum of one-thousand terms each of the value $\frac{2718}{1000} = 2.718$.

Comment: If we insist on writing 2718 as a sum of 1000 positive integers with maximal product of the summands, then writing it as a sum of 282 2s and 718 3s does the trick. Notice that 2 and 3 are the two consecutive integers either side of 2.718.

RESEARCH CORNER

In general, for a positive integer S, the maximal value of $y = \left(\frac{S}{x}\right)^x$ occurs at $x = \frac{S}{e}$. So does the maximal value of $\left(\frac{S}{N}\right)^N$ for an integer N always occur at the first integer larger than $\frac{S}{e}$? Is the matching value of $\frac{S}{N}$ sure to be close to e or, at the very least, sure to be between 2 and 3?

Do you care to explore this dual problem?

Of all the ways to write an integer P as a *product* of N positive integers, which gives the *smallest* possible sum of factors?

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