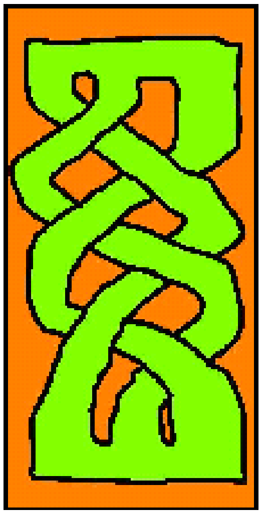

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Whoa! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JUNE 2014

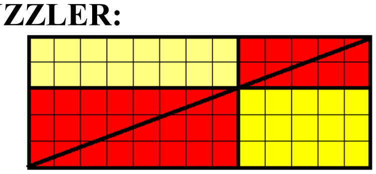
PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

***

Look out for *Primo*: A fun, mathematically rich board game being developed by www.mathforlove.com .
Join their kick-starter campaign at: https://www.kickstarter.com/projects/343941773/primo-the-beautiful-colorful-mathematical-board-ga

ALSO … Join James Key (@iheartgeo) at 9 pm EDT on Tuesday June 10 at the *Global Math Department* for a presentation on factoring: *What is factoring all about really? Should students study it? How can it be seen and developed in a natural and meaningful context for students?* All good mulling!
Sign up at … https://www.bigmarker.com/communities/GlobalMathDept/about

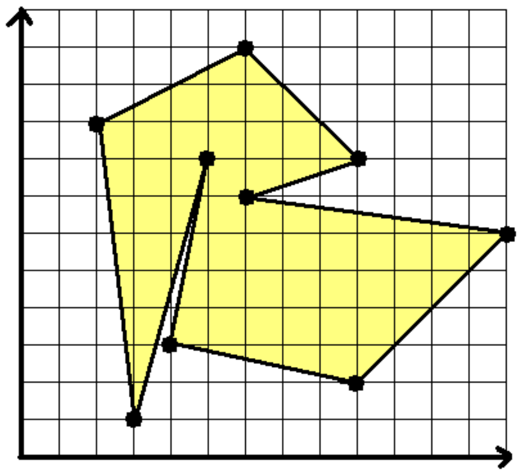
PUZZLER:



16 = 15 ?

COMPUTING AREAS

Here's a nonagon with vertices, in turn, $(2,9)$, $(6,11)$, $(9,8)$, $(6,7)$, $(13,6)$, $(9,2)$, $(4,3)$, $(5,8)$, and $(3,1)$.
What is its area?



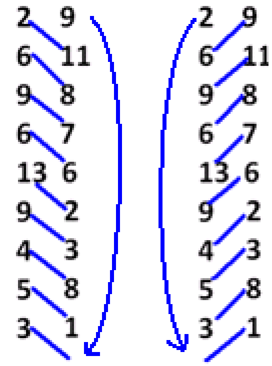
German mathematician and physicist Carl Friedrich Gauss (1777-1855) discovered, among many astounding and profound accomplishments throughout his lifetime, a wondrous procedure for computing the area of a polygon in the coordinate plane using only the coordinates of its vertices.

For our nonagon above the algorithm proceeds as follows:

March around the boundary of the polygon in a consistent direction and list in a column the coordinates of the vertices encountered in turn. This produces two individual columns of numbers: one of the x-coordinates of the vertices encountered, and one with the matching y-coordinates.

2	9
6	11
9	8
6	7
13	6
9	2
4	3
5	8
3	1

Multiply pairs of entries along southeast diagonals (with "wrap around" at the ends) and sum. (Call this Sum 1.) Do this again for south-west diagonals (to get Sum 2). Half the difference of these two sums is the area of the polygon!



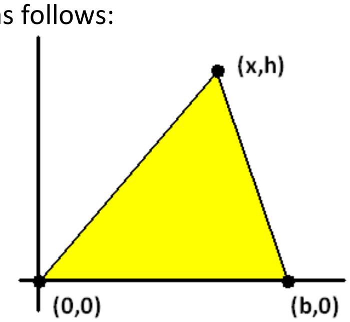
Sum 1
$$= 2\times 11 + 6\times 8 + 9\times 7 + 6\times 6 + 13\times 2$$
$$+ 9\times 3 + 4\times 8 + 5\times 1 + 3\times 9 = 286$$
Sum 2:
$$= 9\times 6 + 11\times 9 + 8\times 6 + 7\times 13 + 6\times 9$$
$$+ 2\times 4 + 3\times 5 + 8\times 3 + 1\times 2 = 395$$

$$Positive\ Difference = 395 - 286 = 109$$

$$Area = \frac{1}{2}\cdot 109 = 54.5.$$

The general formula that results from this procedure is known as Gauss' *shoelace area formula.* (Can you see why it is given this name? Perhaps superimpose the above two diagrams.)

EXAMPLE: A triangle with base b and height h can be situated in the coordinate plane as follows:



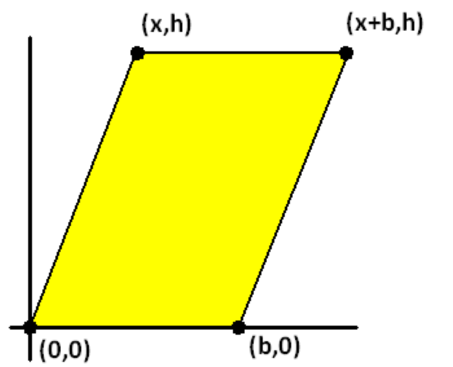
According to the shoelace formula its area is:

$$\begin{array}{cc} 0 & 0 \\ b & 0 \\ x & h \end{array} \qquad \begin{array}{cc} 0 & 0 \\ b & 0 \\ x & h \end{array}$$

$$Sum1 = 0 + bh + 0 = bh$$
$$Sum2 = 0 + 0 + 0 = 0$$

$$Area = \frac{1}{2}(bh - 0) = \frac{1}{2}bh,$$
which is correct!

EXAMPLE: A parallelogram with base of length b and height h can be situated in the coordinate plane as follows:



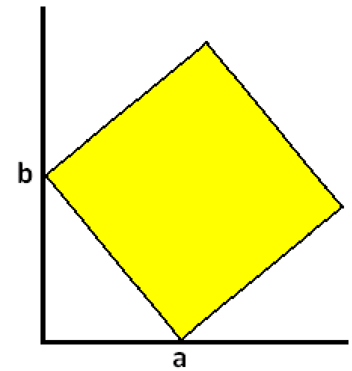
According to the shoelace formula, its area is:

$$\begin{array}{cc} 0 & 0 \\ b & 0 \\ x+b & h \\ x & h \end{array} \qquad \begin{array}{cc} 0 & 0 \\ b & 0 \\ x+b & h \\ x & h \end{array}$$

$$Sum1 = 0 + bh + (x+b)h + 0 = 2bh + xh$$
$$Sum2 = 0 + 0 + xh + 0 = xh$$

$$Area = \frac{1}{2}(2bh + xh - xh) = bh,$$
which is correct!

EXERCISE: A square of side-length c (and hence area c^2) sits in the corner of the first quadrant as shown, touching the x-axis at position a and the y-axis at position b.

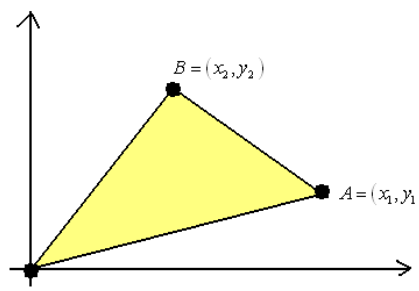


What does the shoelace formula give for the area of this square?

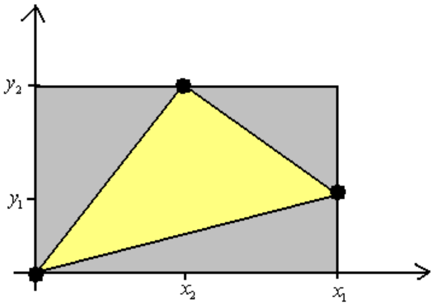
This goal of this essay is to establish why this lovely shoelace algorithm works.

STEP 1: NICELY-SITUATED TRIANGLES

Consider a triangle with one vertex at the origin $O = (0,0)$. Suppose its remaining two vertices are $A = (x_1, y_1)$ and $B = (x_2, y_2)$. What is the area of this triangle?

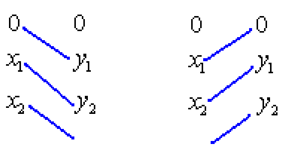


An efficient way to compute this area is to enclose the triangle in a rectangle and subtract from the area of that rectangle the areas of three right triangles.



$$Area = x_1y_2 - \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}(x_1 - x_2)(y_2 - y_1)$$
$$= \frac{1}{2}(2x_1y_2 - x_1y_1 - x_2y_2 - x_1y_2 + x_2y_2 + x_1y_1 - x_2y_1)$$
$$= \frac{1}{2}(x_1y_2 - x_2y_1)$$

This is the shoelace formula applied to the three coordinate points we have:

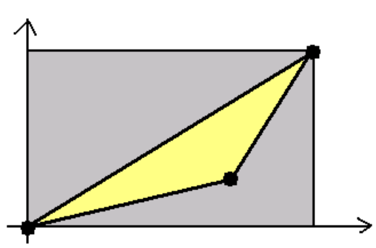


$$Sum1 = 0 + x_1y_2 + 0$$
$$Sum2 = 0 + y_1x_2 + 0.$$

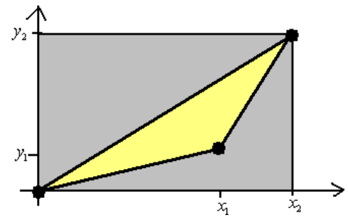
Half their difference: $\frac{1}{2}(x_1y_2 - x_2y_1)$.

So the shoelace formula does match the area of the triangle in this case.

But the diagram we drew does not represent all the ways that the vertices of the triangle could be situated. Consider, for example, three vertices situated as follows:



If we label the vertices this way:

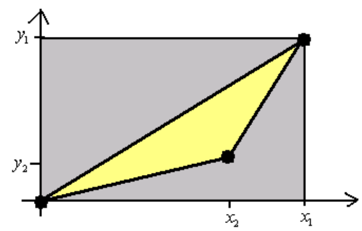


the area of the triangles is:

$$\text{Area } = \frac{1}{2}(x_1y_2 - x_2y_1).$$

(Check this! Subtract the areas of three right triangles and one small rectangle.)

If, instead, we label the vertices as:



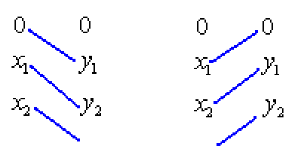
the area of the triangle is now given by:

$$\text{Area } = \frac{1}{2}(x_2y_1 - x_1y_2).$$

These two formulas differ by a minus sign.

Either way, both match half the difference of the two shoelace sums, either

$\frac{1}{2}(Sum1 - Sum2)$ or $\frac{1}{2}(Sum2 - Sum1)$.

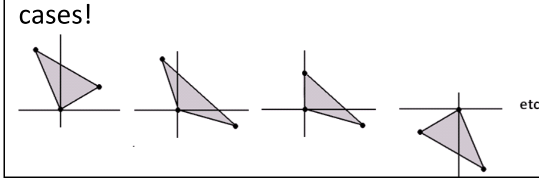


Exercise: For a tedious exercise …

Verify (or argue logically) that one of the shoelace formulas, either

$\frac{1}{2}(Sum1 - Sum2)$ or $\frac{1}{2}(Sum2 - Sum1)$,

matches the area of the triangle no matter in which quadrant or on which axis its two non-origin vertices lie. Check all possible cases!



STEP 2: GENERAL TRIANGLES

Consider a triangle with coordinates

$$A=(x_1,y_1)$$
$$B=(x_2,y_2)$$
$$C=(x_3,y_3)$$

What's its area?

Let's translate the triangle so that one of its coordinates lies at the origin. This will not change the area of the triangle.

Specifically, let's perform the translation that shifts C to the origin. The translated triangle has vertices:

$$O=(0,0)$$
$$A'=(x_1-x_3,y_1-y_3)$$
$$B'=(x_2-x_3,y_2-y_3)$$

Here:

Sum 1 $=(x_1-x_3)(y_2-y_3)$.

Sum 2 $=(x_2-x_3)(y_1-y_3)$.

By step 1, the area of the triangle is half the difference of these sums:

$$\frac{1}{2}\left((x_1-x_3)(y_2-y_3)-(x_2-x_3)(y_1-y_3)\right)$$

(subtracting Sum 2 from Sum 1).

Some algebra shows this is:

$$\frac{1}{2}\left(x_1y_2+x_2y_3+x_3y_1-y_1x_2-y_2x_3-y_3x_1\right)$$

which is the shoelace formula:

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array} \qquad \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}$$

but with the understanding that we could be off by a minus sign. (Maybe the positive difference comes from subtracting Sum 1 from Sum 2 instead.)

STEP 3: BEING CLEAR ON MOTION

We know that the area of a triangle does not change under rigid motions. If the shoelace formula is to match the area of triangles, the formula should be invariant under rigid motions too! (We assumed this was the case for translations in step 2.)

Exercise: Suppose a triangle with vertices:

$$A=(x_1,y_1)$$
$$B=(x_2,y_2)$$
$$C=(x_3,y_3)$$

is translated to the triangle with vertices:

$$A'=(x_1-c,y_1-d)$$
$$B'=(x_2-c,y_2-d)$$
$$C'=(x_3-c,y_3-d)$$

Show that $\frac{1}{2}(Sum1-Sum2)$ is the same for both of these triangles.

original

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array} \qquad \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{array}$$

translation

$$\begin{array}{cc} x_1-c & y_1-d \\ x_2-c & y_2-d \\ x_3-c & y_3-d \end{array} \qquad \begin{array}{cc} x_1-c & y_1-d \\ x_2-c & y_2-d \\ x_3-c & y_3-d \end{array}$$

Exercise: A counter-clockwise rotation about the origin through an angle θ takes a point (x,y) to:

$$(x\cos\theta-y\sin\theta, x\sin\theta+y\cos\theta).$$

Suppose a triangle with vertices:

$$A=(x_1,y_1)$$
$$B=(x_2,y_2)$$
$$C=(x_3,y_3)$$

is rotated to a triangle with vertices

$$A'=(x_1\cos\theta-y_1\sin\theta, x_1\sin\theta+y_1\cos\theta)$$
$$B'=(x_2\cos\theta-y_2\sin\theta, x_2\sin\theta+y_2\cos\theta)$$
$$C'=(x_3\cos\theta-y_3\sin\theta, x_3\sin\theta+y_3\cos\theta)$$

Show that $\frac{1}{2}(Sum1-Sum2)$ is the same for both of these triangles.

By performing the combination of a translation, a rotation about the origin, and another translation, it now follows that the formula $\frac{1}{2}(Sum1 - Sum2)$ is unchanged by arbitrary rotations.

Challenge: How does the formula $\frac{1}{2}(Sum1 - Sum2)$ if the triangle undergoes a reflection?

STEP 4: BEING CLEAR ON SIGNS

The shoelace formula is also invariant under another type of action.

Exercise: Suppose a triangle has vertices:

$$A = (x_1, y_1)$$
$$B = (x_2, y_2)$$
$$C = (x_3, y_3)$$

There are six ways to list the order of these vertices.

Show that in these three orderings:

$$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{matrix} \qquad \begin{matrix} x_3 & y_3 \\ x_1 & y_1 \\ x_2 & y_2 \end{matrix} \qquad \begin{matrix} x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{matrix}$$

the value of $\frac{1}{2}(Sum1 - Sum2)$ is the same, and for the orderings:

$$\begin{matrix} x_3 & y_3 \\ x_2 & y_2 \\ x_1 & y_1 \end{matrix} \qquad \begin{matrix} x_1 & y_1 \\ x_3 & y_3 \\ x_2 & y_2 \end{matrix} \qquad \begin{matrix} x_2 & y_2 \\ x_1 & y_1 \\ x_3 & y_3 \end{matrix}$$

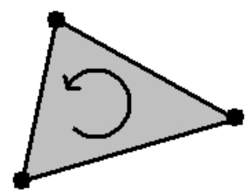
$\frac{1}{2}(Sum1 - Sum2)$ adopts the value with opposite sign.

The previous exercise shows that if one picks a consistent direction around the triangle – either march clockwise (with area always to your right) or march counter-clockwise (with area always to your left)– the shoelace formula $\frac{1}{2}(Sum1 - Sum2)$ is then independent of the choice of starting vertex.

We also know that $\frac{1}{2}(Sum1 - Sum2)$ represents the area of the triangle (if this value turns out to be a positive number) or is the negative of the area.

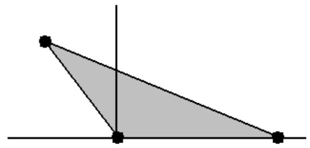
It would be nice to know which direction we should traverse triangles to be sure we obtain the positive area.

RESULT: *If one traverses a triangle in a counter-clockwise direction, then* $\frac{1}{2}(Sum1 - Sum2)$ *is positive, and is the area of the triangle.*



Reason: We can translate the triangle so that one vertex lies at the origin. (This affects neither the area of the triangle nor the value of $\frac{1}{2}(Sum1 - Sum2)$ nor any choice of orientation of the triangle.)

By performing a rotation about the origin we can position the triangle so that one edge lies on the positive x-axis and so that the triangle sits in the upper half-plane. (This rotation also does not affect the area of the triangle, nor the value of the $\frac{1}{2}(Sum1 - Sum2)$, nor any choice of orientation.)



We now have a triangle with vertices, listed in a counter-clockwise order, of the form:

$$(0,0)$$
$$(a,0)$$
$$(c,d)$$

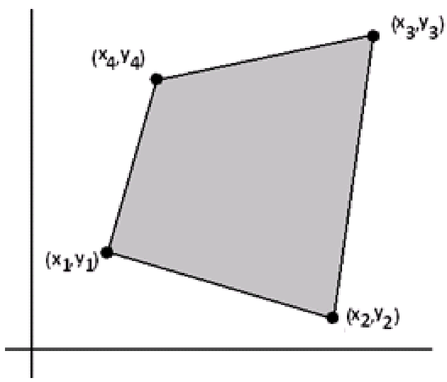
with a and d each positive. Here:

$$\frac{1}{2}(Sum1 - Sum2) = \frac{1}{2}ad$$

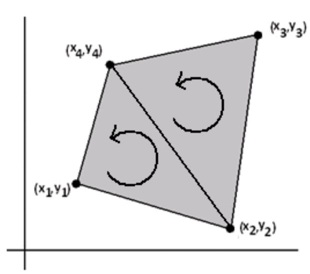
which is indeed positive (and is the area of the triangle).

STEP 5: QUADRILATERALS

Here's a quadrilateral in the coordinate plane:



We can compute its area by subdividing it into two triangles and applying the shoelace formula $\frac{1}{2}(Sum1 - Sum2)$ to each piece, each given the counter-clockwise orientation.



(Notice that if we were to march around the full quadrilateral in a counter-clockwise direction, then that march "induces" counter-clockwise orientations on each of the interior triangles.)

LEFT AREA:

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_4 & y_4 \end{array}$$

$$Area = \frac{1}{2}(x_1y_2 + x_2y_4 + x_4y_1 - y_1x_2 - y_2x_4 - y_4x_1)$$

RIGHT AREA:

$$\begin{array}{cc} x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{array}$$

$$Area = \frac{1}{2}(x_2y_3 + x_3y_4 + x_4y_2 - y_2x_3 - y_3x_4 - y_4x_2)$$

Summing gives:

$$Total\ \ Area = \frac{1}{2}\begin{pmatrix} x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 \\ - y_1x_2 - y_2x_3 - y_3x_4 - y_4x_1 \end{pmatrix}$$

which – delightfully - is the formula $\frac{1}{2}(Sum1 - Sum2)$ applied to the counter-clockwise list of the quadrilaterals vertices:

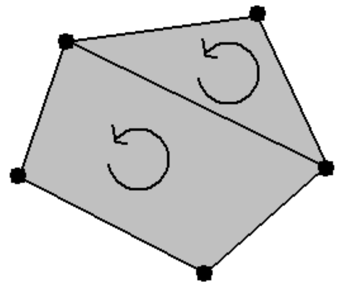
$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{array}$$

(Notice how the terms corresponding to the interior edge (x_2, y_2) to (x_4, y_4) cancel in the sum of left and right areas. This interior edge is traversed twice by the triangles, but in opposite directions.)

RESULT: *List the vertices of a quadrilateral in the coordinate plane in a counter-clockwise order. Then the area of the quadrilateral is given by the shoelace formula:* $\frac{1}{2}(Sum1 - Sum2)$.

STEP 6: BEYOND QUADRILATERALS

If a polygon with N sides can be seen as constructed by adjoining a triangle to one edge of a smaller polygon:



then one can write the shoelace formula $\frac{1}{2}(Sum1 - Sum2)$ for the area of the $(N-1)$-sided polygon:

$$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_{N-1} & y_{N-1} \end{matrix}$$

and the shoelace formula for the area of the triangle:

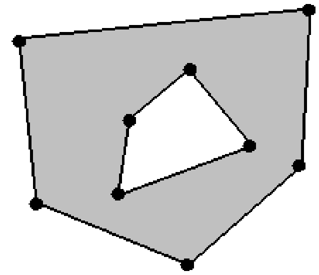
$$\begin{matrix} x_1 & y_1 \\ x_{N-1} & y_{N-1} \\ x_N & y_N \end{matrix}$$

and add the two formulas. One sees that terms corresponding to the vertices of the common interior edge again cancel. The resulting formula is the shoelace formula as applied directly to the original N-gon!

(Notice again that if we follow a counter-clockwise orientation for the large polygon, all interior polygons naturally follow a counter-clockwise orientation as well.)

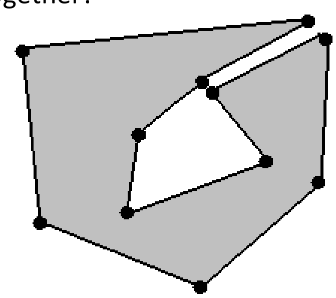
This establishes that Gauss shoelace formula works for all reasonable polygons.

CHALLENGE: Does the formula work for a polygon with a hole?



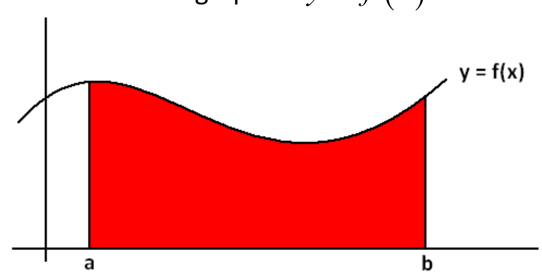
Can some version of it be made to work?

Comment: One can, of course, compute the area of the outer polygon and subtract from it the area of the hole. But can one, in any reasonable way, interpret this holey polygon as a non-holed polygon with two parallel congruent edges brought infinitely close together?

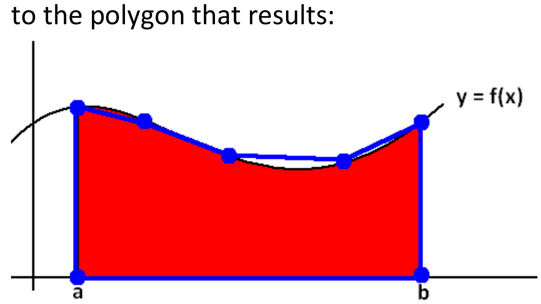


Is this way of thinking helpful? Interesting?

QUERY: Suppose we wish to compute the area under the graph of $y = f(x)$ shown:



We could approximate the curve via line segments and apply the shoelace formula to the polygon that results:



By taking the limit of finer and finer approximations, we should approach the true area under the curve: $\int_a^b f(x)\,dx$.

Does the shoelace formula give us a new way to think about integrals?

EXERCISE: In the opening example of a nonagon we followed a clockwise orientation, not a counter-clockwise path. Could this have been a problem?

RESEARCH CORNER:
Is there a three-dimensional version of the shoelace formula, one that computes volumes of polyhedra?

tanton.math@gmail.com