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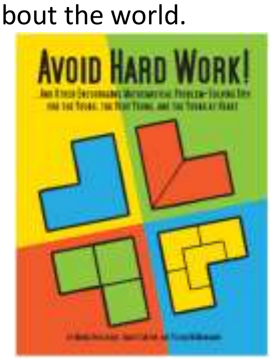
WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

July 2016

A NEW PROBLEM-SOLVING BOOK!

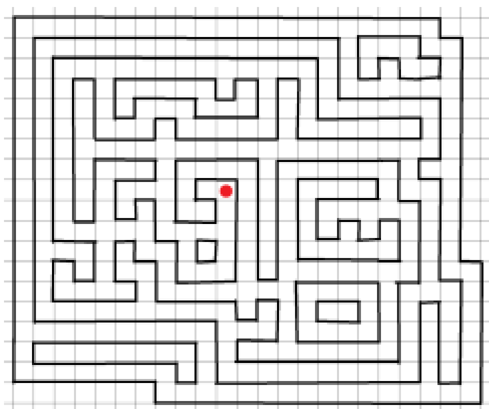
Authors Maria Droujkova, Yelena McManaman, and I have combined forces to take the ten problem-solving strategies from my Curriculum Inspirations work for the MAA and translate those ideas into spectacularly fabulous, accessible, and practical mathematical work for the young, the very young, and the young at heart. Learn how to teach – and engage in - joyous mathematical thinking right from the get-go. Teach how to play mathematically and to reason and be curious about the world.



But first the book needs to be published! And you can help us do that. Check out: http://naturalmath.com/avoid-hard-work/ . Your support will be so much appreciated!

THIS MONTH'S PUZZLER:
This puzzle is inspired by one of the activities discussed in the book!

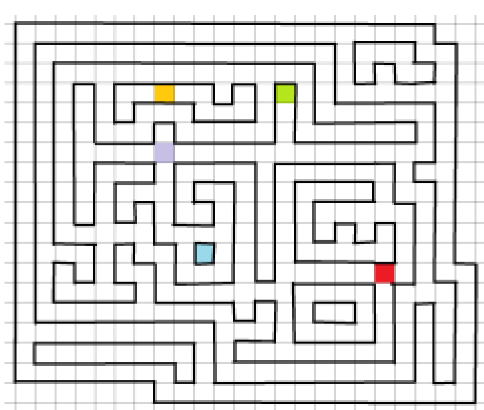
On graph paper I've drawn a picture of an island. On this island there are lakes, which contain islands, which contain lakes, and so on.



Is the red dot on land or in water? What is the easiest way to answer questions like these?

ANOTHER PUZZLER ABOUT ISLANDS AND LAKES (AND ISLANDS WITHIN LAKES WITHIN ISLANDS WITHIN LAKES!)

The diagram in the puzzler looks like a maze of paths enclosed in a region. Each cell in the region is surrounded by 0,1,2,3, or 4 walls (as demonstrated by the mauve, red, mustard, green, and blue cells, respectively).



The majority of cells have 2 walls, and we might expect this to be the average value of the number of walls each cell in the region possesses.

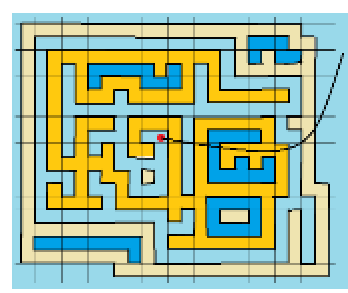
What's your gut feeling here? Do you think that the average count of walls for cells in diagrams like these to be 2, right on the nose, every single time? Or just under this value? Or just over? Or does it depend on the diagram drawn?

[Just to be clear, these diagrams are composed of unit line segments – "walls" – connecting neighboring grid points. There is one outer loop of walls, and each and every grid point within that loop has precisely two walls attached to it.]

Aside Question: *If one constructs an outer loop first, will one always be able to connect its interior grid points appropriately?*

THE ANSWER TO THE FIRST PUZZLER

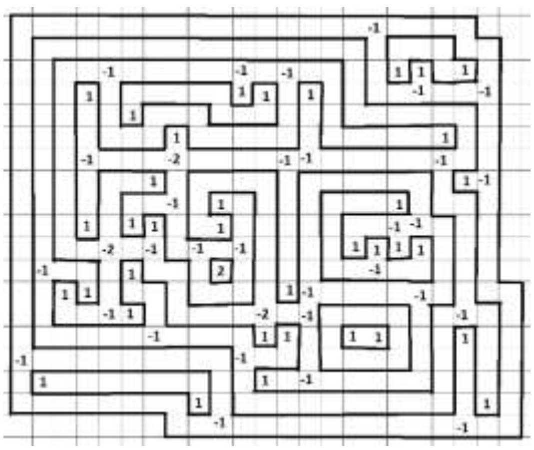
If we just color the land and the water we see that the dot is in the water. (I used here two different shades of yellow and two different shades of blue to highlight the four bodies of land and six bodies of water.)



But that's hard! An easier approach is not to color and instead just follow a line, of any shape, to the dot from the anywhere in the outside ocean. Each time we cross a boundary line, we are either moving from water to land or from land to water. As the line I happened to draw crosses a boundary line eight times - an even number - the dot must be in water. No coloring needed. (By the way, this means that any path, no matter how loopy, from the outside region to the dot must cross a boundary line an even number of times.)

TOWARDS THE SECOND PUZZLER:

Let's pretend that each cell has, on average, precisely two walls, and go through the diagram labelling each cell in it with the count of "excess walls" it possesses. For example, the blue cell has four walls, so label it 2 for having two extra walls above the supposed average. The red cell, with just one wall, will be labeled -1 as it has an "excess" of negative one walls from the supposed average. We won't bother labeling the cells with precisely two walls. (They would be labeled 0.)



Now sum all the numbers you see. In this diagram, the numbers sum to $+2$. Ahh! This sum is not zero and so the average number of walls per cell is not two.

Here's a bold claim.

THEOREM: *In all such diagrams labeled this way the sum of labels is sure to be* $+2$.

If this is true, then the average count of walls surrounding each cell is strictly greater than two for all diagrams we can possibly draw.

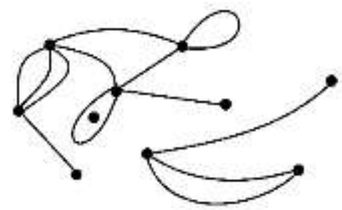
[Did your intuition suggest this too?]

PROVING THE CLAIM

To prove the claim we need a classic result from graph theory.

Here a *graph* is simply a collection of dots drawn on a page with lines (curved or straight) connecting pairs of dots. The dots are usually called *vertices* and the lines *edges*. We permit more than one edge to connect the same pair of dots and we allow an edge to connect a vertex to itself.

The picture shows a graph with $v = 10$ vertices and $e = 13$ edges.



Also, this graph has $c = 3$ components: it comes in three disjoint pieces (with one of the pieces a single vertex).

Notice too that this graph has no intersecting edges: when edges meet they do so only at their endpoints at a vertex. A graph drawn this way is called a *planar graph.*

> **Challenge**: Not all graphs are planar. Can you <u>prove</u> that the graph with five vertices and ten edges, one edge between each and every pair of vertices, is not planar?

Our example graph also divides the white space of the page into regions. It has $r = 6$ finite regions (and there is the large, outer, infinite region of white space too, which we shall not count).

The classic result from graph theory is:

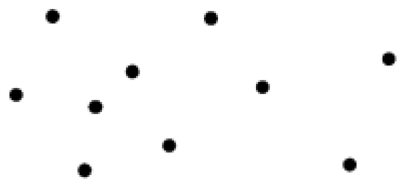
For all planar graphs we have

$$v - e + r = c.$$

This is certainly true for our example: $10 - 13 + 6$ does indeed equal 3.

The gist of the proof is straightforward.

The equation is certainly valid for a collection of v vertices just drawn on the page with no edges. Here $e=0$, $r=0$, and $c=v$.



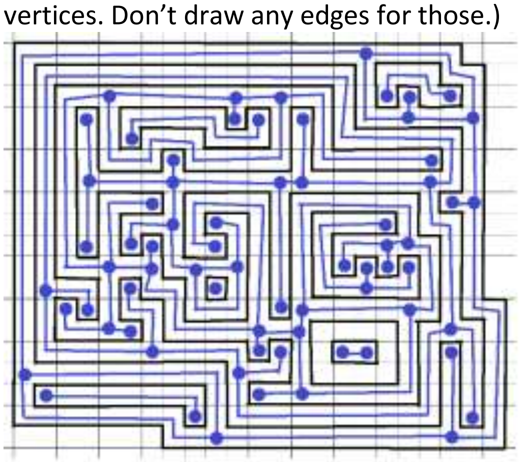
Now add edges in one at a time.

We start with the equation $v-e+r=c$ holding true and we're increasing the value of e by one, in stages.

At each stage, a new edge will either connect two separate components (thereby decreasing the value of c by one and thus keeping the equation $v-e+r=c$ valid) or will start and end on the same component (thereby creating a new region, increasing the value of r by one and thus keeping the equation $v-e+r=c$ valid).

The relationship $v-e+r=c$ thus never fails to hold!

Our picture of islands and water is really a graph. Regard each cell with either 0, 1, 3, or 4 walls as a vertex (that is, each cell we labeled earlier with a number is a vertex) and regard the "corridors" of cells with precisely two walls connecting pairs of vertices as the edges. (There could be corridors that trace a loop and connect no vertices. Don't draw any edges for those.)



Each vertex is of one of four types, depending on the number of edges emanating from it. Let F be the count of vertices with four edges emanating from them (previously our mauve cells with 0 walls), T the count of vertices with three (previously our red cells with just one wall), D the count of cells with one edge emanating from it ("dead ends" – previously our green cells with three walls), and S the count of vertices with no edges (previously our blue "squares" with four walls).

The quantity $4F+3T+1D+0S$ (four for each F cell, three for each T cell, and so on) counts the total number of ends of edges. As each edge has two ends, this means $4F+3T+D=2e$. Also, the total number of vertices is $v=F+T+D+S$.

From $v-e+r=c$ we get:

$$F+T+D+S-2F-\frac{3}{2}T-\frac{1}{2}D+r=c,$$

that is,

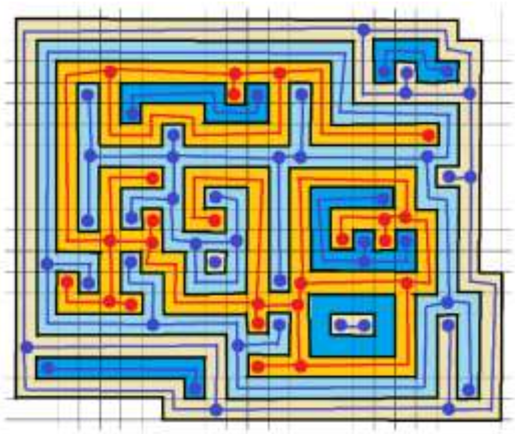
$$2S+D-T-2F=2c-2r.$$

The left side of this equation is the sum of the labels we placed in the islands and lakes picture: we set each S cell as worth 2, each D cell as worth 1, each T cell as worth -1, and each F cell as worth -2, and summed all these labels. (What a coincidence!) And the theorem we cited claims that this sum equals two.

So to prove the theorem, we need to show that $2c-2r=2$, or, equivalently, that $c=r+1$. Is this so?

Why yes!

Our graph is such that each of its components corresponds to a body of land or a body of water. (Look at the component highlighted in red, for instance.)



And each finite region defined by a component is given by a loop that surrounds a lake or an island (which might also have lakes and islands within them).

So "c" counts the total number of bodies of land and water, and "r" counts all the bodies of water or land that are surrounded by land or water, respectively. And that's every lake and every body of land in the diagram except the outermost body of land. So we do indeed have $c = r + 1$, proving the theorem.

Phew!

Comment: These pictures of islands and lakes remind me of the swirly pictures of finger-print and hand-print patterns. Forensic scientists look for *loops* and *deltas* in these patterns, features that looks like vertices with only one edge and vertices with three edges. Our result says something interesting about the topology of our prints! Care to research this?

RESEARCH CORNER

The island and lakes pictures discussed here are constrained to fit on a square grid with a rigid "two walls per grid point" criterion.

A general picture of islands and lakes is much more freeform.

Is there a natural way to associate a graph with such a free diagram of loops nested within loops? Will there always seem to be natural points in the picture that "want" a particular number of edges emanating from them? (Like the ones in blue below, for example.)

Is there some sum of counts that is invariant for all these diagrams?

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