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WHAT HO! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

JULY 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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MATHFEST! It's a biggie! The centennial celebration of the MAA.
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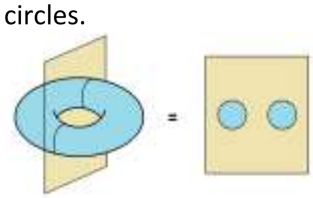
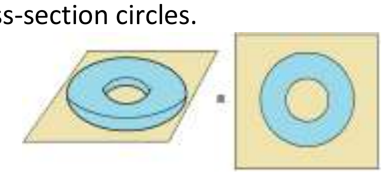
OPENING PUZZLE:
One can slice a donut in a vertical plane to obtain a cross-section slice of two perfect circles.



A horizontal slice also produces two cross-section circles.

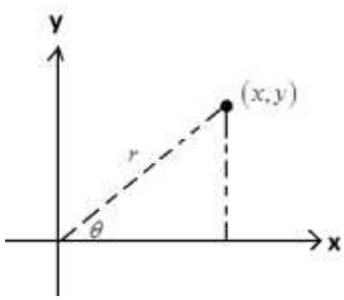


Is there a third way to slice a donut to produce two perfect circles on the cross-section plane?

The two ways to slice a bagel to see (ideally) two circles in the cross-section slice of each half.

ROTATING POINTS ABOUT AXES

Any point (x, y) in the plane can be written in the form $(r\cos\theta, r\sin\theta)$.



$$x = r\cos\theta$$
$$y = r\sin\theta$$

The point (X,Y) obtained by rotating the point α degrees counterclockwise about the origin is $(r\cos(\theta+\alpha), r\sin(\theta+\alpha))$. Using

$$\cos(\theta+\alpha) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$$
$$\sin(\theta+\alpha) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$$

we see that

$$X = \cos\alpha x - \sin\alpha y$$
$$Y = \sin\alpha x + \cos\alpha y.$$

We can extend this work to three dimensions. Suppose we rotate a point (x, y, z) about the z-axis through an angle α. It's z-coordinate will not change, but its x- and y-coordinates do. If the coordinates of the rotated point are (X,Y,Z), then:

$$X = \cos\alpha x - \sin\alpha y$$
$$Y = \sin\alpha x + \cos\alpha y$$
$$Z = z.$$

If we rotate the point through an angle α about the y-axis, we then have:

$$X = \cos\alpha x - \sin\alpha z$$
$$Y = y$$
$$Z = \sin\alpha x + \cos\alpha z.$$

Rotating about the x-axis gives instead the point with coordinates

$$X = x$$
$$Y = \cos\alpha y - \sin\alpha z$$
$$Z = \sin\alpha y + \cos\alpha z.$$

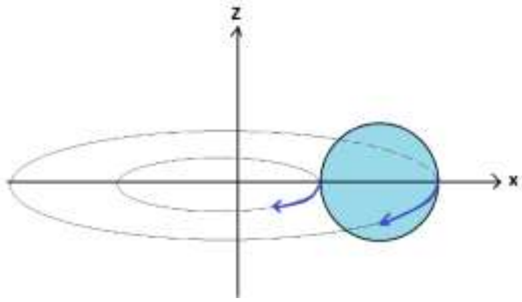
Comment: In this third example, since a rotation of $-\alpha$ takes (X,Y,Z) to (x, y, z), we must have:

$$x = X$$
$$y = \cos(-\alpha)Y - \sin(-\alpha)Z$$
$$= \cos\alpha Y + \sin\alpha Z$$
$$z = -\sin\alpha Y + \cos\alpha Z.$$

The inverse relations for the other two cases are analogous.

THE EQUATION OF A TORUS

The official mathematical name for the (surface) shape of a donut is *torus*. The surface is obtained by rotating a circle in the xz-plane about the z-axis.

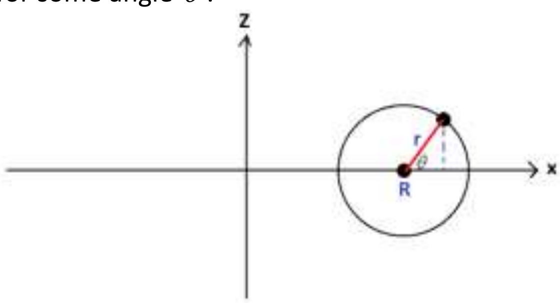


We have means to write down the equation of a torus.

Suppose the center of the circle we rotate is R units from the z-axis and that the circle has radius r. (Assume $r < R$.) Then any point on the circle (still in the xz-plane) has coordinates (x, z) given as

$$x = R + r\cos\theta$$
$$y = 0$$
$$z = r\sin\theta$$

for some angle θ.



Each point on the torus is obtained by rotating one of these points on the circle about the z-axis through some angle $0 \le \alpha < 360^\circ$. It thus has coordinates:

$$X = \cos\alpha\, x - \sin\alpha\, y = \cos\alpha\left(R + r\cos\theta\right)$$
$$Y = \sin\alpha\, x + \cos\alpha\, y = \sin\alpha\left(R + r\cos\theta\right)$$
$$Z = z = r\sin\theta.$$

(These are the parametric equations of a torus.)

Using $\sin^2\alpha + \cos^2\alpha = 1$, we see that $X^2 + Y^2 = \left(R + r\cos\theta\right)^2$ and so $r\cos\theta = \pm\sqrt{X^2 + Y^2} - R$. With $r\sin\theta = Z$ we get

$$\left(\pm\sqrt{X^2 + Y^2} - R\right)^2 + Z^2 = r^2.$$

That's the equation of a torus! (This is akin to $X^2 + Y^2 + Z^2 = r^2$ being the equation of a sphere.)

Comment: The choices $\pm$ each give half the torus. (Which halves?)

SLICING A TORUS

We introduced capital letters X, Y, and Z just to help us keep track of the images of rotated points. To make things look more familiar, let's go back to using small letters x, y, and z in the work that follows (and use X, Y, and Z again later when we need to keep track of another set of image points!).

We have the equation of a torus

$$\left(\pm\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2.$$

The effect of a vertically slicing of the torus can be seen by setting $y = 0$. (The slicing plane is then the xz-plane.) The equation of the surface becomes

$$\left(\pm\sqrt{x^2 + 0^2} - R\right)^2 + z^2 = r^2.$$

This corresponds to two equations:

$$(x - R)^2 + z^2 = r^2$$
$$(-x - R)^2 + z^2 = r^2,$$

that is,

$$(x - R)^2 + z^2 = r^2$$
$$(x + R)^2 + z^2 = r^2.$$

We have two circles in the xz-plane, each with radius r and centers $(R,0)$ and $(-R,0)$, just as expected.

The effect of a horizontal slice (using the xy-plane) can be found by setting $z=0$. This gives the equation:

$$\left(\pm\sqrt{x^2+y^2}-R\right)^2+0^2=r^2,$$

that is,

$$\sqrt{x^2+y^2}=R+r \text{ or } R-r \text{ or } r-R \text{ or } -r-R.$$

Only two of these possibilities are meaningful:

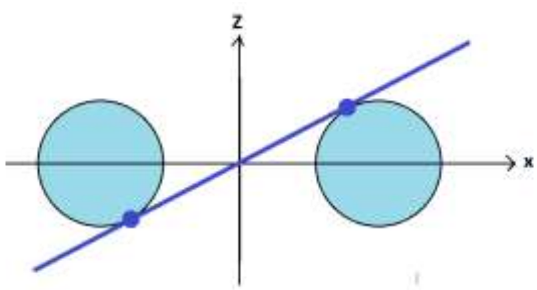
$$x^2+y^2=(R+r)^2$$

$$x^2+y^2=(R-r)^2$$

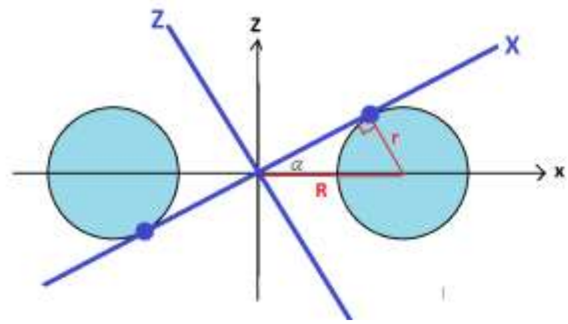
and we have two concentric circles of the radii we expect.

THE OPENING PUZZLE

There is indeed a third planar cut that yields two circles in the cross-section. It is given by the diagonal cut indicated in this diagram:



To see this, it is easiest to work in a titled coordinate system. Let's call the blue line the X - axis, the perpendicular axis shown the Z -axis, and we'll set the Y -axis the same as the original y -axis (into the page).



Let α be the angle of tilt of the X -axis as shown. Notice that $\sin\alpha=\frac{r}{R}$ (and so $\cos\alpha=\frac{\sqrt{R^2-r^2}}{R}$.)

Since the xyz-coordinate system is a rotation of the XYZ-system about the common Y-axis through and angle $-\alpha$, any point (X,Y,Z) in the XYZ-system has coordinates (x,y,z) given by

$$x=\cos\alpha X+\sin\alpha Z$$

$$\mathrm{y}=\mathrm{Y}$$

$$\mathrm{z}=-\sin\theta X+\cos\alpha Z$$

in the xyz-coordinate system.

Since points in the xyz-system lying on the torus satisfy the equation

$$\left(\pm\sqrt{x^2+y^2}-R\right)^2+z^2=r^2,$$

points in the XYZ-system lie on the torus if

$$\left(\pm\sqrt{(\cos\alpha X+\sin\alpha Z)^2+Y^2}-R\right)^2+(-\sin\alpha X+\cos\alpha Z)^2=r^2.$$

Now let's look at the cross-section on the diagonal XY-plane. Its effect can be found by setting $Z=0$ into this equation. We have

$$\left(\pm\sqrt{\cos^2\alpha X^2+Y^2}-R\right)^2+\sin^2\alpha X^2=r^2.$$

This yields

$$X^2+R^2\pm 2R\sqrt{\cos^2\alpha X^2+Y^2}=r^2.$$

Using $\cos\alpha=\frac{\sqrt{R^2-r^2}}{R}$, this can be rewritten:

$$\pm 2\sqrt{\left(R^2-r^2\right)X^2+Y^2}=X^2+R^2-r^2.$$

Squaring yields:

$$4\left(R^2-r^2\right)X^2+4R^2Y^2 =\left(X^2+Y^2+R^2-r^2\right)^2.$$

Some very tedious algebra shows that this equation can be rewritten

$$\left(X^2+(Y-r)^2-R^2\right)\left(X^2+(Y+r)^2-R^2\right)=0.$$

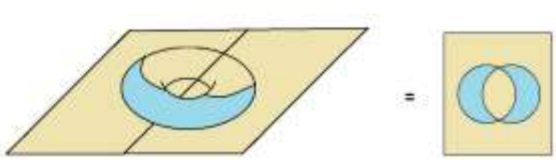
This means that either

$$X^2+(Y-r)^2=R^2$$

or

$$X^2+(Y+r)^2=R^2.$$

We have two circles each of radius R in the diagonal plane with centers $(0,r)$ and $(0,-r)$.



Challenge: Actually slice a donut or a bagel along a diagonal to see these two intersecting congruent circles.
(I've never succeeded, by the way! All baked items I've worked with have been too irregular to see believable intersecting circles.)

RESEARCH CORNER

The torus is a surface with the property that for any point P on the surface there are four distinct surface circles that pass through P. (Can you see the four circles?) Prove that there are not five or more circles passing through a surface point P.

Prove that if a smooth surface has the property that four surface circles pass through each point on the surface, then the surface must be a torus.

A sphere (and the plane, a sphere of infinite radius) has the property that for each point P on the surface that there are infinitely many surface circles that pass through P. Does this property characterize spheres?

Are there surfaces embedded in three-dimensional space with precisely two surface circles passing through any given surface point? Three circles? One?

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