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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

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THIS MONTHS' PUZZLER:

The circle with center $(0,0)$ and radius 1, given by the equation $x^2+y^2=1$, passes through infinitely many points in the plane with each coordinate a rational number. It passes through the points $\left(\frac{3}{5},\frac{4}{5}\right)$ and $\left(-\frac{120}{169},\frac{119}{169}\right)$, for instance. In fact, for any Pythagorean triple $a^2+b^2=c^2$, the points $\left(\pm\frac{a}{c},\pm\frac{b}{c}\right)$ are "rational points" on this circle.

The rational points densely fill the entire plane and intuition suggests that every circle in the plane will pass through infinitely many of these points.

a) Find the equation of a circle whose graph passes through no rational points of the plane. (That is, find the equation of a circle for which no point (x,y) with both x and y rational satisfies the equation.)

b) Show that the circle of radius 2 and center $\left(\sqrt{2},\sqrt{2}\right)$ passes through only one rational point, namely, $(0,0)$.

c) Find an equation of a circle whose graph passes through the points $(-1,0)$ and $(1,0)$ and no other rational points.

Parts a), b), and c) show that it is possible to have circles in the plane that pass through precisely 0, 1, and 2 rational points.

d) Prove that if a circle passes through 3 rational points that it actually passes through infinitely many rational points.

RATIONAL POINTS ON CIRCLES

Consider the circles with center $(0,0)$. These satisfy an equation of the form $x^2+y^2=r^2$ for some radius r.

If we seek a point (x,y) with x and y each rational satisfying this equation, then it better be the case that r^2 is rational too.

Thus there can be no rational points on the circle with equation $x^2+y^2=\sqrt{2}$ of radius $r=\sqrt{\sqrt{2}}$, for instance.

Challenge: We just observed that if r^2 is irrational, then there are no rational points on the circle with equation $x^2+y^2=r^2$. But the converse need not be true: If r^2 is rational, it is still possible that the circle $x^2+y^2=r^2$ might not pass through any rational points.

Prove, for instance, that there are no rational points on the circle with equation $x^2+y^2=3$.

HINT: Assume there is a rational point. Write the fractions in each coordinate of this point over a <u>least</u> common denominator: $\left(\frac{a}{d},\frac{b}{d}\right)$. From $a^2+b^2=3d^2$ we see that a^2+b^2 must be a multiple of three. Argue that this is problematic!

Challenge: Deduce that if a circle with equation $x^2+y^2=N$ for some integer N has a rational point, then N must equal the sum of two square numbers. (Which integers are the sum of two squares?)

Consider the circle of radius 2 and center $\left(\sqrt{2},\sqrt{2}\right)$. It has equation

$$\left(x-\sqrt{2}\right)^2+\left(y-\sqrt{2}\right)^2=4.$$

Certainly $(x,y)=(0,0)$ satisfies this equation. Our challenge to is to prove no other rational points do.

The equation can be rewritten

$$x^2+y^2=2\sqrt{2}\left(x+y\right).$$

Suppose there are two rational values x and y satisfying this equation.

First observe that if $x+y\neq 0$ we'd then be able to write

$$\sqrt{2}=\frac{x^2+y^2}{2\left(x+y\right)},$$

suggesting that $\sqrt{2}$ is rational. This is not so, and so it must be that $x+y=0$. In this case the equation reads $x^2+y^2=0$, forcing $x=y=0$. Thus $x=0$, $y=0$ is the only rational point on this circle.

Consider the circle with center $\left(0,\sqrt{2}\right)$ of radius $\sqrt{3}$. It has equation $x^2+\left(y-\sqrt{2}\right)^2=3$ for which both $x=1, y=0$ and $x=-1, y=0$ are solutions.

From $x^2+y^2-1=2\sqrt{2}y$ we see that if y is rational and not zero, x cannot be rational. (Otherwise, $\sqrt{2}=\frac{x^2+y^2-1}{2y}$ would be a rational expression for $\sqrt{2}$.) Thus $(-1,0)$ and $(1,0)$ are the only two rational points on this circle.

Zero, One, Two, Many

Suppose a circle with equation $(x-a)^2+(y-b)^2=r^2$ passes through three rational points: (p_1,p_2), (q_1,q_2), and (s_1,s_2). We have then that

$$p_1^{\,2}-2p_1a+p_2^{\,2}-2p_2b=s_1^{\,2}-2s_1a+s_2^{\,2}-2s_2b$$
$$q_1^{\,2}-2q_1a+q_2^{\,2}-2q_2b=s_1^{\,2}-2s_1a+s_2^{\,2}-2s_2b$$

which is a system of two linear equations in the unknowns a and b with all coefficients rational.

$$\alpha a+\beta b=\gamma$$
$$\delta a+\varepsilon b=\phi$$

Since we know the circle has a center, this system of equations will have a solution.

Solving this system will give values for a and b that are rational combinations of its rational coefficients. We conclude a and b are each rational numbers.

From $(p_1-a)^2+(p_2-b)^2=r^2$ it follows that r^2 is rational too.

Any circle passing through three rational points has center that is also a rational point and a radius whose square is rational.

By translating the circle, let's assume the center of the circle is the rational point $(0,0)$. (All rational points on the circle will be translated to rational points too.)

Choose one of the rational points on this circle. Let's call it (m,n) this time. We have

$$m^2+n^2=r^2.$$

For any rational number λ, consider the line through (m,n) of slope λ. Points on this line are given by $(m+t, n+\lambda t)$ for some parameter t.

Let's determine where this line intercepts the circle a second time. One checks that from $(m+t)^2+(n+\lambda t)^2=r^2$ we deduce $t=-\frac{2m+2\lambda n}{1+\lambda^2}$. It follows then that

$$m+t=\frac{\left(\lambda^2-1\right)m-2\lambda n}{1+\lambda^2}$$
$$n+\lambda t=\frac{-2\lambda m+\left(\lambda^2-1\right)n}{1+\lambda^2}$$

gives the coordinates of another rational point on the circle. So, for each choice of rational value λ we have a new rational point on the circle.

Any circle centered at the origin passing through one rational point actually passes through infinitely many rational points.

Translating back to our original circle with center (a,b), we conclude:

Any circle passing through three rational points actually passes through infinitely many rational points.

RESEARCH CORNER

Is it possible for a square in the plane to pass through no rational points? Just one? Precisely two?

Is it possible for an equilateral triangle in the plane to pass through no rational points? Just one? Precisely two?

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