CURRICULUM INSPIRATIONS: www.maa.org/ci

Innovative Online Courses: www.gdaymath.com
Tanton Tidbits: www.jamestanton.com

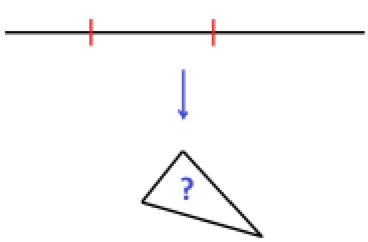
WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

January 2017

THIS MONTH'S PUZZLER
It's a classic – and it is problematic!

1. A straight stick is broken, at random, into three pieces. What are the chances those pieces form a triangle of positive area?



2. Suppose instead a stick is broken at random into two pieces and then the right piece is broken at random again. What are the chances that the three pieces obtained this way form a triangle?

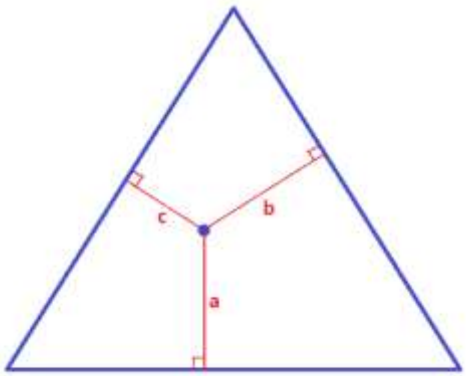
3. A stick is broken at random into two pieces and then the longer piece is broken at random. What are the chances that the three pieces obtained form a triangle?

4. A stick is broken at random into two pieces, and a coin is tossed to decide which piece is broken again. What are the chances that the three pieces obtained form a triangle?

Are these four questions the same?

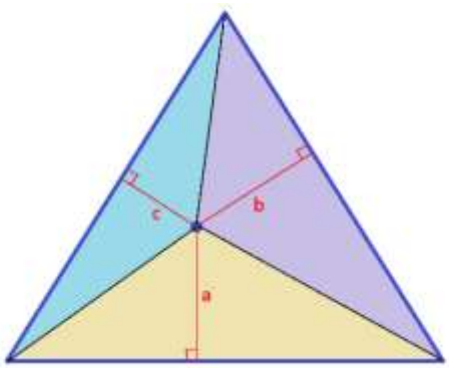
VIVIANI'S THEOREM

Italian scholar Vincenzo Viviani (1622 – 1703) noted that for any point inside an equilateral triangle, the sum of its distances from each of the three sides is constant. That constant is the height of the triangle.

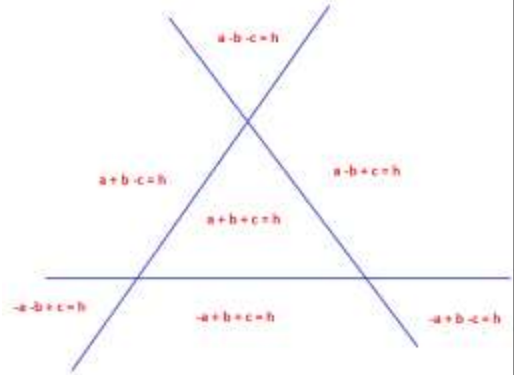


a + b + c = height

If we call the side length of the triangle is s and the height of the triangle h, then computing the area of the triangle as the sum of three individual triangle areas gives $\frac{1}{2}sa+\frac{1}{2}sb+\frac{1}{2}sc=\frac{1}{2}sh$. Viviani's claim then follows.



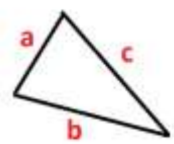
Challenge: If we allow points to sit outside the triangle, with distances to the sides of the triangle recorded as negative in a sum as appropriate, then the entire plane is divided into regions of invariant sum values.



Can you see why this is so? (One can work with triangle areas here too.)

BROKEN STICKS AND TRIANGLES

Suppose a stick, of length 1, say, is broken into three pieces: a left piece of length a, a middle piece of length b, and a right piece of length c (with $a+b+c=1$).



The three pieces form a triangle of positive area if and only if the three triangular inequalities hold.

$$\begin{aligned} a+b>c &\quad \Rightarrow 1=a+b+c>2c \\ a+c>b &\quad \Rightarrow 1=a+b+c>2b \\ b+c>a &\quad \Rightarrow 1=a+b+c>2a \end{aligned}$$

This is equivalent to requiring each piece to have length less than $\frac{1}{2}$.

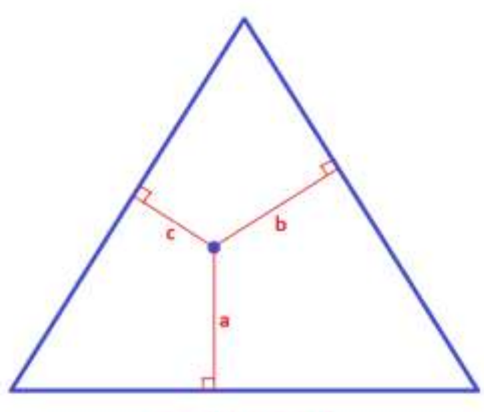
The opening puzzler first asks: *If a stick of length* 1 *is randomly broken into three* *pieces, what are the chances that each piece has length less than* $\frac{1}{2}$*?*

Probability problems like these are notoriously difficult. They are usually ill-defined and one can argue most any number answer as valid and correct! The problem is that, typically, the random process utilized is not spelled out, thus making the puzzle ambiguous.

For example, my brain, right off the bat, thinks of three different ways one might choose and record two break points in a stick.

METHOD 1:
We are looking for a way to select an ordered triple of numbers (a,b,c) with $a+b+c=1$.

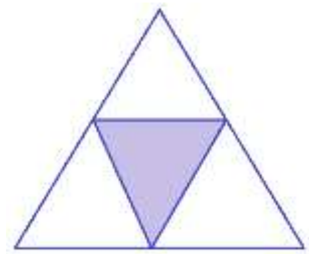
If we draw an equilateral triangle of height 1, then Viviani's theorem shows that each such triple corresponds to a unique point inside the triangle.



This suggests a method for breaking the stick into randomly chosen pieces.

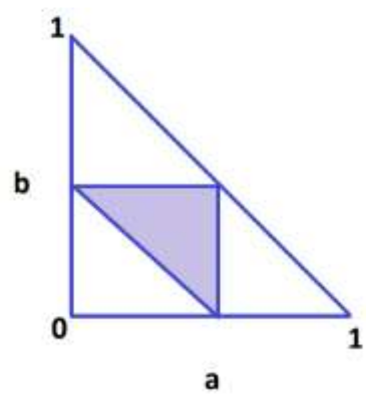
Throw a dart at such an equilateral. Measure the distances a*,* b*, and* c *from where the dart lands and break the stick into left, middle, and right pieces of these lengths in turn.*

As the three lengths form the sides of a triangle if and only if each has value less than $\frac{1}{2}$, only if the dart lands in the middle quarter of the equilateral triangle will they form a triangle. The probability of seeing this is $\frac{1}{4}$.



METHOD 2:
When I think of actually breaking a stick, I would probably break off a left piece first (that is, randomly – uniformly – choose a value a in the interval from 0 to 1), and then break off a middle piece (randomly choose a value b in the interval from 0 to $1-a$). The remainder of the stick will have length $c=1-a-b$.

With this process, our "parameter space" is the set of all ordered pairs (a,b) with $0<a<1$ and $0<b<1-a$. This defines a region of the plane that is a right triangle with each point inside the triangle corresponding to a way to break the stick.



A dart thrown at this triangle gives a point that corresponds to a broken stick yielding a triangle if and only if it lands a point with

$a < \frac{1}{2}$, $b < \frac{1}{2}$, and $c = 1 - a - b < \frac{1}{2}$, that is, if and only if the chosen point lies inside the shaded region shown. This too is $\frac{1}{4}$ of the area.

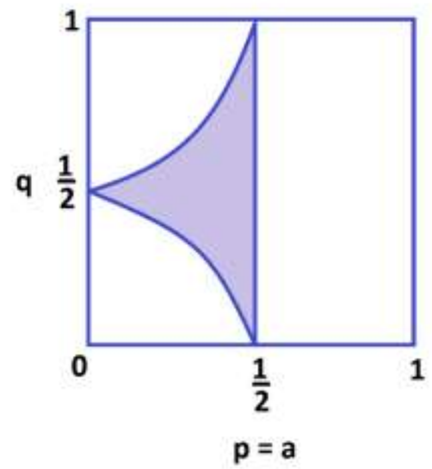
METHOD 3:
Actually, another natural way to break a stick into three pieces is to begin by choosing two proportion values: first a value p in $(0,1)$ and then another value q in $(0,1)$. Then let a be the proportion p of the full stick and b be the proportion q of what remains. (For example, I could decide to chop off 20% of the stick, and then 30% of what remains; or first chop off 83% of the stick and then 22% of what remains.) This gives the three lengths

$$a = p \times 1 = p$$
$$b = q(1-a) = q(1-p)$$
$$c = 1 - p - q(1-p).$$

Here we have a parameter space of points (p,q) in a unit square. Such a point corresponds to a broken stick with pieces that form a triangle if and only if each of a, b, and c is less than $\frac{1}{2}$. These give the constraints

$$p < \frac{1}{2}$$
$$q < \frac{1/2}{1-p}$$
$$q > 1 - \frac{1/2}{1-p},$$

yielding the shaded region shown.



This shaded region has area

$$\int_0^{\frac{1}{2}} \frac{1/2}{1-p} - \left(1 - \frac{1/2}{1-p}\right) dp$$
$$= \int_0^{\frac{1}{2}} \frac{1}{1-p} - 1 dp = \ln 2 - \frac{1}{2}.$$

Thus using a dart thrown at this diagram for a random method of breaking the stick yields pieces that make a triangle about 19.3% of the time.

Something unsettling
In case you feel uncomfortable with this third answer, consider this question.

A stick is broken, at random, into three pieces: a left piece, a middle piece, and a right piece. What are the chances that the left piece has length less than $\frac{1}{2}$*?*

Intuitively, we might expect the answer to be 50%: the left break point is equally likely to be to the left or to the right of the halfway mark. (And details of the second break point seem irrelevant to the size of this left piece.)

This is assuming we are following a procedure of choosing a left and then a right break point one after the other a la methods 2 or 3. If we assume that both break points occur simultaneously (that the stick has already been broken somehow and is currently hiding under the table out of view), then we might say that the approach of method 1 could be appropriate.

In any case, three-quarters of the points of the equilateral triangle of method 1 have $a < \frac{1}{2}$, three-quarters of the points of the right triangle of method 2 have $a < \frac{1}{2}$, and one-half of the points of the unit square have $p = a < \frac{1}{2}$. Of our three methods, only method three matches our intuitive answer of 50%.

Challenge: Consider the question:

A 1 meter stick has been broken into a left piece, a middle piece, and a right piece. You are shown the left piece and see that it has length less than $\frac{1}{2}$. What do you say are the chances that all three pieces form a triangle?

FOCUS ON THE LEFT PIECE

Each of our three methods of choosing a stick breaking pattern involve selecting a point at random in a diagram: throw a dart either at an equilateral triangle, a right triangle, or a square. There are other means to select points randomly in such diagrams.

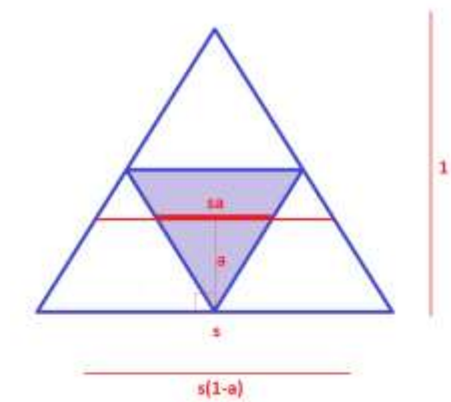
Here's another.

Select a value at random between 0 and 1 and call it a. Now, in a given diagram, draw a line segment that has that constant a value. Select a point at random along that line segment. You now have a randomly selected point in the diagram.

This approach has you first select a, the length of the left piece of the stick, and then has you determine the lengths of the remaining two pieces.

And now we shall see something remarkable: with this approach, all three methods are consistent!

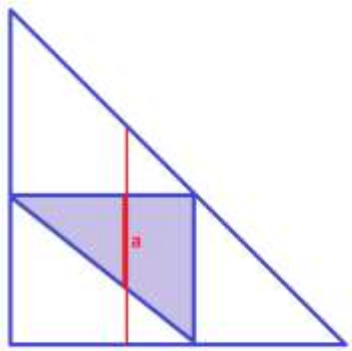
In method 1, a line of constant a-value is a horizontal segment of length $s(1-a)$. (Play with similar triangles.)



Let P_a be the probability that a point chosen along this line segment lands in the region that gives a broken stick with pieces that form a triangle. Similar triangles show

$$P_a = \begin{cases} 0 & \text{if } a > \frac{1}{2} \\ \frac{sa}{s(1-a)} = \frac{a}{1-a} & \text{if } a < \frac{1}{2}. \end{cases}$$

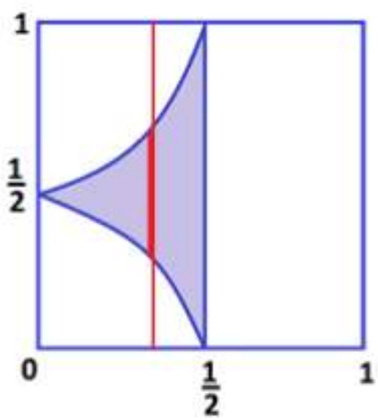
In method 2, a line of constant a value is a vertical segment of length $1-a$. Similar triangles show that if $a < \frac{1}{2}$, then the length of this vertical segment that lies inside the center shaded triangle is a.



This method 2 also gives

$$P_a = \begin{cases} 0 & \text{if } a > \frac{1}{2} \\ \frac{a}{1-a} & \text{if } a < \frac{1}{2}. \end{cases}$$

In method 3, a line of constant a value is a vertical segment of length 1.



We too see that

$$P_a = \begin{cases} 0 & \text{if } a > \frac{1}{2} \\ \frac{1/2}{1-a} - \left(1 - \frac{1/2}{1-a}\right) = \frac{a}{1-a} & \text{if } a < \frac{1}{2}. \end{cases}$$

And these formulae match our intuitive approach too!

Suppose we are given piece of the stick of length a. What are the chances P_a that the second piece will be broken to yield three pieces that make a triangle?

Well, if $a > \frac{1}{2}$, we have no hope.

If $a < \frac{1}{2}$, then we need the break point of the second piece of length $1-a$ give two pieces each of length less than $\frac{1}{2}$ as well. This means its break point is to lie within this region of length a along the length $1-a$.

The chances of this occurring are $\frac{a}{1-a}$.

(We get this same answer if we think of choosing a breakpoint along the length of the segment directly or by selecting a proportion $0 < p < 1$.)

SO IN SUMMARY …

A stick is broken into three pieces. We are told that the left piece has length a. Then the chances that all three pieces form a triangle given this information are

$$P_a = \begin{cases} 0 & \text{if } a > \frac{1}{2} \\ \frac{a}{1-a} & \text{if } a < \frac{1}{2}. \end{cases}$$

Moreover, this answer is consistent for all the modes of thinking we have presented.

Question: Are the chances the same if we are told that it is the right piece has length a? How about if we are just told that one of the end pieces has length a?

What if we are told the middle piece has length a?

What if we are told Just that one of the pieces has length a?

Summing Probabilities.

Suppose I told you that I am going to break off the left end of the stick at a length of either 0.1 m, or 0.2 m, … , 0.9 m, or 1.0 m, each option equally likely. If the remainder of the stick is broken at random, what are the chances that the three pieces obtained will create a triangle?

Well, in playing this game multiple times, we'd have a $P_{0.1}$ chance of "winning" one tenth of the time, $P_{0.2}$ chance of winning one tenth, of the time, and so on. This gives the total chance of winning

$$P = P_{0.1} \times \frac{1}{10} + P_{0.2} \times \frac{1}{10} + \cdots P_{1.0} \times \frac{1}{10}.$$

If I said I was going to break off the left end at one of the lengths 0.01 m, 0.02 m, …, 1.00 m, each equally likely, then the chances of winning this game would be

$$P = P_{0.01} \times \frac{1}{100} + P_{0.02} \times \frac{1}{100} + \cdots P_{1.00} \times \frac{1}{100}$$

and so on.

Taking this process to the limit, we see that the probability of obtaining three pieces of stick that form a triangle, given that we are handed a left piece of some length a m ,is

$$P = \int_0^1 P_a \, da \, .$$

We can compute this.

$$\begin{aligned} P &= \int_0^{1/2} \frac{a}{1-a} da \\ &= \int_0^{1/2} \frac{1}{1-a} - 1 \, da \\ &= \left[\ln|1-a| - a \right]_0^{1/2} \\ &= \ln 2 - \frac{1}{2}. \end{aligned}$$

ANSWERS

In all the ways we've looked at matters, we've obtained the same answer for puzzle 2, namely, $\ln 2 - \frac{1}{2} \approx 19\%$.

The answer to the third puzzle is double this. (We'd be taking to the limit sums of the form

$$\begin{aligned} P &= P_{0.1} \times \frac{1}{5} + \cdots + P_{0.5} \times \frac{1}{5} \\ &= 2\left(P_{0.1} \times \frac{1}{10} + \cdots + P_{0.5} \times \frac{1}{10} \right). \end{aligned}$$

Alternatively, we can argue that half the time for puzzle 2 we'd be in the situation of puzzle 3.)

In puzzle 4, half the time we have no hope of seeing a triangle and half the time we're in the situation of puzzle 3. The answer here is back to $\ln 2 - \frac{1}{2}$. (Alternatively, if handed an end piece of stick we might as well deem it to be the left piece and be in the situation of puzzle 2.)

And as for puzzle 1 … I have no answer.

RESEARCH CORNER

I've been very careful throughout this essay to speak of a left piece, a middle piece, and a right piece of stick. Did I need to be careful? To what extent does order or pieces matter for our analysis?

Develop other means for "breaking a stick at random" that yield multiple, valid, answers to question 1. Can every real number between zero and one be obtained as an answer to the question?

tanton.math@gmail.com